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Asymptotic stability of large scale dynamical systems using computer generated Lyapunov functions

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ASYMPTOTIC STABILITY OF LARGE SCALE DYNAMICAL SYSTEMS USING COMPUTER GENERATED LYAPUNOV FUNCTIONS

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I. INTRODUCTION

Stability is a property which must be considered in control system design. Whatever the control system performance criterion may be, one must check that the resulting system is stable with respect to disturbances and initial deviations from its state of rest.

The concept of stability originates in mechanics where it characterizes the equilibrium point of a rigid body. An equilibrium point of a rigid body is considered stable if the body remains close to its original position after being moved slightly from its position of rest, and it is considered asymptotically stable if it returns to its original equilibrium.

Nonlinear differential equations representing dynamical systems are generally so complex that they cannot be solved analytically in a closed form. Lyapunov stability theory is one of the qualitative approaches which is concerned with the behavior of families of solutions of a given differential equation and which does not seek explicit solutions.

It is generally difficult to apply Lyapunov stability results to high-dimensional systems with complicated structures. This difficulty lies in the fact that there

is no universal and systematic procedure available which
tells us how to find the required Lyapunov functions.
Although converse Lyapunov theorems have been established,
these results provide no clue, except in the case of
linear equations, for the construction of Lyapunov
functions.

Brayton and Tong [1], [2] established a constructive
algorithm for generating Lyapunov functions to analyze the
stability and the global asymptotic stability of dynamical
systems. These Lyapunov functions will be used in the
present dissertation as the basis for the stability
analysis of large scale dynamical systems.

The domain of attraction of an equilibrium, sometimes
known as the asymptotic stability region of an
equilibrium, is the region which has the property that
trajectories of the system starting within it will
eventually approach the equilibrium point.

An efficient algorithm was developed in [3] to
estimate the domain of attraction of an equilibrium of a
nonlinear dynamical system, using the constructive
procedure of [1] and [2]. Also, a method of analyzing
complex systems in terms of lower order subsystems was
presented in [4] to circumvent the difficulties that
usually arise in the application of the results of [1] and

[2] to high dimensional systems with intricate structures. The results in [4] yield conditions for global asymptotic stability.

A method of estimating the domain of attraction of an equilibrium of an interconnected system is presented in this dissertation, using the results of [3] and [4]. An interconnected system is decomposed into several sub-systems. The stability regions of isolated subsystems are determined using the constructive algorithm developed in [3]. A test matrix for the overall system is then generated using parameters obtained from the stability analysis of the isolated subsystems and from the interconnection characteristics of the system. If this matrix is an M-matrix, then the equilibrium of the system is guaranteed to be asymptotically stable in some region. The largest stability region of the overall system is found by using an optimally weighted sum of Lyapunov functions. The components of this sum are Lyapunov functions for the isolated subsystems of the interconnected system.

The above results are applied to a multi-machine power system [5] to find its stability region and to estimate the critical clearing time of a faulted power system.

It is also shown that Lyapunov functions can be used in stabilizing a long train control system [6] traveling on steep grades. A straightforward application of optimal control theory and observer theory to this problem is either not possible or it may require a computational effort that is impractical. The large number of state variables of the long train prohibits the direct evaluation of the feedback matrix. Using the results of [4], this system is stabilized by feedback, such that the system is globally asymptotically stable.

In summary, this dissertation is concerned with the stability analysis of an equilibrium of large scale dynamical systems, with emphasis on the stability analysis of a multi-machine power system and a train control system, using computer generated Lyapunov functions.

II. NOTATION

Let U and V be arbitrary sets. If u is an element of
U, we write u ε U. If U is a subset of V, we write
U $\subset$ V and we denote the boundary of U by ∂U. Union of
sets is denoted by $\bigcup$. If W is a convex polyhedral
region, then E[W] denotes the set of extreme points of W,
and H[W] represents the convex hull of a set W. Supremum,
infimum, maximum, and minimum are denoted by sup, inf,
max, and min, respectively.

Let R denote the real line, let R^+ = [0,∞), and let
R^n denote n-dimensional Euclidean space. For the vectors
x, yεR^n the notation x $\leqslant$ y is used to indicate each
component of x is less than or equal to the corresponding
component of y. The symbol $|\cdot|$ denotes a vector norm on
R^n. The symbol $\|\cdot\|$ is used to denote the matrix norm
induced by some vector norm. If f is a function or
mapping of a set X into a set Y, we write f:X$\rightarrow$Y.

Matrices are assumed to be real and we denote them by
upper case letters. If A = [a_{ij}] is an arbitrary matrix,
then A^T denotes the transpose of A. An eigenvalue of a
square matrix A is identified as $\lambda(A)$ and Re $\lambda(A)$ denotes
the real part of $\lambda(A)$. For a square matrix A, the
inverse, if it exists, is denoted by A^{-1}. We call a real

(n x n) matrix $A = [a_{ij}]$ an M-matrix if $a_{ij} < 0$ for all $i \neq j$ and if all principal minors of A are positive. The symbol I is used to denote an identity matrix.

The time derivative of a variable (e.g., dx/dt) is expressed by a dot over the variable (e.g., $\dot{x}$). If $v: R^n \to R$, then $\nabla v(x)$ denotes its gradient and $\nabla v(x)^T$ is the transpose of the gradient. We use $Dv_{(E)}(x)$ to denote the total time derivative of $v(x)$ along the trajectories of a dynamical system described by differential equations of the form

$$\dot{x} = f(x). \tag{E}$$

A comparison function $\phi: R^+ \to R^+$ belongs to class K (i.e., $\phi \in K$) if $\phi(0) = 0$ and if $\phi(r_1) < \phi(r_2)$ whenever $r_1 < r_2$. If $\phi \in K$ and if, in addition, $\lim_{r \to \infty} \phi(r) = \infty$, then we say ϕ belongs to class KR (i.e., $\phi \in KR$). We define a ball $B(r)$ by $\{x \in R^n : |x| < r\}$ for some $r > 0$.

III. STABILITY THEORY

The purpose of this chapter is to present the background material required in subsequent chapters to understand the stability analysis of dynamical systems described by ordinary differential equations.

In section A, the First Method and the Second Method of Lyapunov will be presented. Since the literature on these two methods is enormous (see, e.g., [7], [8]), only the basic results will be presented, without proofs.

In section B, an algorithm developed by Brayton and Tong [1], [2] is presented. This algorithm is used to construct Lyapunov functions for dynamical systems. Brayton and Tong use the notion of an asymptotically stable set of matrices to determine the asymptotic stability of an equilibrium of dynamical systems.

In section C, an efficient procedure [3] is presented to estimate the domain of attraction of an equilibrium point of a system which is locally asymptotically stable.

The results in [1]-[3] are significant and powerful, but they are not applicable to high dimensional systems. In Chapter IV, these difficulties will be removed to a certain extent by viewing a system, whenever possible, as an interconnection of several lower order subsystems, and

following the approach employed in [9], by analyzing such systems in terms of their isolated subsystems and in terms of the system interconnecting structures.

In this chapter, and in subsequent chapters, systems described by ordinary differential equations of the form

$$\dot{x} = f(x) \tag{E}$$

are considered, where $x \epsilon R^n$, $\dot{x} = dx/dt$, $t \epsilon R^+$, and $f: B(r) \to R^n$ for some $r > 0$. Henceforth, it is assumed that f is sufficiently smooth so that (E) possesses, for every $x_o \epsilon B(r)$ and for every $t_o \epsilon R^+$, one and only one solution $x(t; x_o, t_o)$ for all $t > t_o$, where $x_o = x(t_o; x_o, t_o)$. x_o is called an initial point, t is referred to as "time", and t_o is called initial time. Henceforth, it is assumed that (E) admits the trivial solution $x = 0$ so that $f(0) = 0$ for all $t > t_o$. This solution is also called an equilibrium or a singular point of (E). In addition, it is assumed that $x = 0$ is an isolated equilibrium, i.e., there exists $r' > 0$ so that $f(x') = 0$ holds for no nonzero $x' \epsilon B(r')$.

The preceding formulation pertains to local results. When discussing global results, it is always assumed that $f: R^n \to R^n$ and that f is sufficiently smooth, so that (E) possesses, for every $x_o \epsilon R^n$ and for every

$t_o \epsilon R^+$, a unique solution $x(t;x_o,t_o)$ for all $t > t_o$. In this case, it is also assumed that $x=0$ is the only equilibrium of (E).

A. Lyapunov Stability Theory

Since (E) cannot generally be solved analytically in a closed form, the qualitative properties of the equilibrium $x=0$ are of great practical interest. This motivates the following stability definitions in the sense of Lyapunov.

Definition 1. The equilibrium $x=0$ of (E) is <u>stable</u> if, for every $\epsilon > 0$ and any $t_o \epsilon R^+$, there exists a $\delta > 0$ such that $|x(t;x_o,t_o)| < \epsilon$ for all $t > t_o$ whenever $|x_o| < \delta$.

Definition 2. The equilibrium $x=0$ of (E) is <u>asymptotically</u> <u>stable</u> if (i) it is stable, and (ii) there exists a $\delta_1 > 0$ such that $\lim_{t \to \infty} x(t;x_o,t_o) = 0$ whenever $|x_o| < \delta_1$.

The set of all $x_o \epsilon R^n$ such that condition (ii) of definition 2 is satisfied is called the <u>domain of attraction</u> of the equilibrium $x=0$ of (E).

Definition 3. The equilibrium $x=0$ of (E) is <u>globally asymptotically</u> <u>stable</u> if it is stable and if every solution of (E) tends to zero as $t \to \infty$.

The stability analysis of nonlinear systems is, in general, very complex. It is, however, reasonable to expect that the stability criteria for linear systems could be applied to nonlinear systems if the deviations from the equilibrium state are sufficiently small, so that the nonlinearity has only a minor effect. Each equilibrium point, if there are more than one, is investigated separately. If an equilibrium point is not at the origin, it can be transferred to the origin by an appropriate coordinate transformation.

Theorem 1. The equilibrium x=0 of (E) is asymptotically stable if the Jacobian matrix J(x), evaluated at x=0, has only eigenvalues with negative real parts.

(I.e., $J = \left[\frac{\partial f(x)}{\partial x}\right]_{x=0}$, the Jacobian matrix evaluated at x=0, has only eigenvalues with negative real parts.)

We call the equation $\dot{x}=Jx$ the first approximation (or the linearized system) to the nonlinear system (E). If at least one eigenvalue of the matrix J has a positive real part, then the equilibrium x=0 of (E) is unstable. Theorem 1 is called the Indirect Method of Lyapunov. It is also known as the First Method of Lyapunov in the literature.

The Direct Method of Lyapunov, sometimes known as the Second Method of Lyapunov, is used to ascertain the

stability properties of an equilibrium of dynamical system (E) without explicit knowledge of its solutions. This method has its origin in energy considerations. Lyapunov's idea was to generalize an energy argument by introducing energy-like scalar functions and computing their rate of change with respect to time along the motions of the system under consideration.

Let $v(x)$ be a scalar function of x. Then the sign definiteness of $v(x)$ and of the total time derivative of $v(x)$ along the solutions of (E), given by

$$Dv_{(E)}(x) = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}(x) \cdot f_i(x) = \nabla v(x)^T \cdot f(x),$$

will determine the stability of an equilibrium $x=0$ of (E), where $x_1, \cdots, x_n$ and $f_1, \cdots, f_n$ are respectively the components of x and f. The function v which is assumed to be continuously differentiable with respect to all its arguments, is referred to as Lyapunov function in the literature. In the following, such Lyapunov functions will be characterized as being positive definite, negative definite, radially unbounded, and positive (negative) semidefinite.

Definition 4. A function v is said to be positive definite if there exists $\phi \varepsilon K$ such that $v(x) \geq \phi(|x|)$ for all $x \varepsilon B(r)$ for some $r>0$, and if $v(0) = 0$.

Definition 5. A function v is said to be _negative definite_ if -v is positive definite.

Definition 6. A function v, defined on R^n, is said to be _radially unbounded_ if there exists $\phi \in KR$ such that $v(x) > \phi(|x|)$ for all $x \in R^n$ and if $v(0)=0$.

Definition 7. A function v is said to be _positive (negative) semidefinite_ if $v(x) > 0$ $(v(x) < 0)$ for all $x \in B(r)$ for some $r > 0$, and $v(0)=0$.

In the following, we let $\Omega \subset R^n$ be a domain and we assume that the equilibrium x=0 of (E) is in its interior. The next three theorems exemplify results from the Direct Method of Lyapunov.

Theorem 2. For all $x \in \Omega$, if there exists a positive definite function $v(x)$ such that $Dv_{(E)}(x)$ is negative semidefinite, then the equilibrium x=0 of (E) is stable.

Theorem 3. If $v(x)$ is positive definite and if $Dv_{(E)}(x)$ is negative definite on Ω, then the equilibrium x=0 of (E) is asymptotically stable.

In the next theorem, we assume that $v: R^n \to R$.

Theorem 4. For all $x \in R^n$, if there exists a positive definite and radially unbounded function $v(x)$ such that $Dv_{(E)}(x)$ is negative definite, then the equilibrium x=0 of (E) is globally asymptotically stable.

Let Ω and v be as given in Theorem 3. It can be shown that if a domain D defined by

$$D = \left\{ x \varepsilon R^n : v(x) < d, d > 0 \right\}$$

is entirely contained in Ω, then this domain D will be contained in the domain of attraction of x=0 of (E). If, in particular, d>0 is the largest constant such that $D \subset \Omega$ is true, then D will be the largest estimate of the domain of attraction which can be obtained by this method for the specific Lyapunov function v(x).

B. Constructive Stability Results

In [1] and [2], Brayton and Tong present an algorithm used to construct Lyapunov functions. They use these functions to establish the stability and the global asymptotic stability of the equilibrium x=0 of (E). We summarize here some global results from [1] and [2]. Please refer to [1] and [2] for further details.

By rewriting the equation (E): $\dot{x} = f(x)$ as

$$\dot{x} = M(x)x \qquad\qquad (E')$$

where M(x) is chosen so that $M(x)x = f(x)$ for all $x \in R^n$, and applying Euler's formula to (E'), the following difference equation is obtained:

$$x_{k+1} = x_k + h_k M(x_k)x_k \qquad (1)$$

where $h_k = t_{k+1} - t_k$ and $k = 0,1,2,\cdots\cdot$. For every $x_k \in R^n$, $M(x_k)$ will be a real $(n \times n)$ matrix. Let S denote the set of all matrices obtained by varying x_k over all allowable values. Then (1) can be rewritten equivalently as

$$x_{k+1} = (I_n + h_k M_k)x_k \quad , M_k \in S \qquad (2)$$

where I_n denotes the $(n \times n)$ identity matrix. In [1] and [2] it is shown that if the equilibrium $x = 0$ of (2) is stable (globally asymptotically stable) for all sequences $\{h_k\}, 0 < h_k < h'$ for some $h' > 0$, then the equilibrium $x = 0$ of (E) is stable (globally asymptotically stable). The above result can be restated in an equivalent way which makes use of the stability properties of a class of matrices. Let S denote the set of $(n \times n)$ real matrices with the property that for every $x \in R^n$ there exists an $M \in S$ such that $f(x) = Mx$. Suppose that the set A of $(n \times n)$ matrices given by

$$A = \{I_n + hS\} \qquad\qquad (3)$$

is stable (asymptotically stable) for some $h > 0$. (The precise definitions of these two terms are given in the next paragraphs.) Then, the equilibrium $x=0$ of (E) is stable (globally asymptotically stable).

We call a set A of $(n \times n)$ real matrices stable if, for every neighborhood of the origin $U \subset R^n$, there exists another neighborhood of the origin $V \subset R^n$ such that for every $M \epsilon A'$, $MV \subseteq U$. Here, A' denotes the multiplicative semigroup generated by A, and

$$MV = \{u \epsilon R^n : u = Mv, \; v \epsilon V\}.$$

In [1] it is shown that for a class of stable matrices the following statements are equivalent:

a) A is stable

b) A' is bounded

c) There exists a bounded neighborhood of the origin $W \subset R^n$ such that $MW \subseteq W$ for every $M \epsilon A$. Furthermore, W can be chosen to be convex and balanced.

d) There exists a vector norm $|\cdot|_W$ such that $|Mx|_W < |x|_W$ for all $M \epsilon A$ and for all $x \epsilon R^n$.

Now let $\alpha \varepsilon R$ and let $W \subset R^n$. Let $\alpha W = \{u \varepsilon R^n: u = \alpha\, w,$
$w \varepsilon W\}$. Since the above statements c) and d) are related
by [10]

$$\|x\|_W = \inf \{\alpha: \alpha > 0,\ x \varepsilon \alpha W\},$$

it follows that $\|x\|_W$ defines a Lyapunov function for A,
that is, a function v with the property $v(Mx) < v(x)$ for all
$M \varepsilon A$ and $x \varepsilon R^n$.

We call a set of matrices A _asymptotically stable_ if
there exists a number $\rho > 1$ such that ρA is stable. The set
ρA is obtained by multiplying every member of A by ρ. In
[2], it is shown that the following statements are
equivalent:

    a)   A is asymptotically stable.

    b)   There exist a convex, balanced, and

           polyhedral neighborhood of the origin W

           and a positive number $r < 1$ such that for

           each $M \varepsilon A$, $MW \subseteq rW$.

    c)   A is stable and there exists K such that

           for all $M \varepsilon A'$, $|\lambda(M)| < K < 1$.

Note that if A is stable, then rA is asymptotically stable
for all positive $r < 1$.

In [1] and [2], a constructive algorithm is given to determine whether a set of m (n x n) real matrices $A = \{M_0, \cdots, M_{m-1}\}$ is stable by starting with an initial polyhedral neighborhood of the origin W_0 and by defining a sequence of regions W_{k+1} by

$$W_{k+1} \triangleq H\left[\bigcup_{j=0}^{\infty} M_k^j, W_k\right], \quad k' = (k-1) \mod m$$

where $H[\cdot]$ denotes the convex hull of a set.

In [1] and [2], it is shown that A is stable if and only if

$$W^* = \bigcup_{k=0}^{\infty} W_k$$

is bounded. Note that W^* is also given by

$$W^* = H\ [\bigcup MW_0, \ M\epsilon A'].$$

Since all extreme points z of W_{k+1} are of the form $z = M_i^j u$, where u is an extreme point of W_k, we need only deal with the extreme points of W_k in order to obtain

$$W_{k+1} = H\ [\bigcup M_k^j, \ u: \ u\epsilon E(W_K)].$$

Clearly, the new extreme points $E(W_{k+1})$ are images of $E(W_k)$. If $|\lambda(M_{k'})| < 1$ for $M_{k'} \epsilon A$, then there exists an integer $J_{k'}$ such that

$$H \left[\bigcup_{j=0}^{\infty} M_{k'}^{j}, W_k \right] = H \left[\bigcup_{j=0}^{J_{k'}} M_{k'}^{j}, W_k \right].$$

Thus, W_{k+1} will be formed in a finite number of steps, since W_k is expressed as the convex hull of a finite set of points.

A set of matrices A is said to be _unstable_ if A is not stable. In [1], the following instability criterion is established: A is unstable if there exists a k such that $\partial W_o \cap \partial W_k = \phi$, where ϕ denotes the null set. In [2], it is also shown that if a set A of matrices, with E(A) finite, is asymptotically stable, then the constructive algorithm given above will terminate "stable" in a finite number of steps. It can be shown that A is asymptotically stable by showing that ρA is stable for $\rho > 1$ by using the constructive algorithm.

The set A given in (3) consists in general of infinitely many matrices. However, the following result, established in [1], reduces the stability analysis to a

finite set of matrices: let A be a set of matrices in the linear space of (n x n) matrices and let E[A] be the set of extreme matrices of A [11]. Then H[A] is stable if and only if E[A] is stable. Thus, if E[A] is finite, the asymptotic stability analysis of a set of matrices A can be accomplished in a finite number of steps.

C. Domain of Attraction

Many practical systems possess more than one equilibrium point. In such cases, the concept of global asymptotic stability is no longer applicable and one is usually interested in knowing the extent of the domain of attraction of an asymptotically stable equilibrium. In this section, an algorithm [3] is presented to establish a procedure for determining an estimate of the domain of attraction of x=0 of the autonomous system

$$\dot{x} = f(x). \tag{E}$$

By linearization of (E) about the equilibrium point x=0, we obtain

$$\dot{x} = Jx + f_1(x),$$

where $J = \left[\frac{\partial f}{\partial x}(x)\right]_{x=0}$ denotes the Jacobian matrix

evaluated at x=0 and $f_1(x) = f(x) - Jx$. Note that

$$\lim_{|x| \to 0} \frac{|f_1(x)|}{|x|} = 0.$$ If the real parts of the

eigenvalues of J are negative, then the equilibrium x=0 of

the system

$$\dot{x} = Jx \tag{4}$$

is globally asymptotically stable. Furthermore, the

equilibrium x=0 of (E) is locally asymptotically stable.

We apply Euler's method to (4) to obtain the difference

equation

$$x_{k+1} = (I + h_k J)x_k \tag{5}$$

where h_k denotes current computation step size. We now

form for some h>0 an infinite set of matrices defined by

$$A = \left\{ I + h_k J: \ 0 < h_k < h \right\}.$$

Then, the set of extreme matrices of A contains two

matrices, namely, $E(A) = \left\{I, I + hJ\right\}$. Since the identity

matrix I is stable, we need to concern ourselves only

with the singleton $\{I + hJ\}$, where h is determined so that $|\lambda(I + hJ)| < 1$.

We use the constructive algorithm in section B of this chapter to determine a convex set W^* from an initial convex set W_0 using the multiplicative semigroup of the set $\{\rho(I + hJ)\}$, for some $\rho > 1$. For any initial point in W^*, a solution for (5) will approach the origin with time $t \to \infty$ and the same observation is true for (4). Since $x=0$ of (4) is globally asymptotically stable, the above observation is true even if we multiply the extreme vertices of W^* by some constant c, $0 < c < \infty$. Thus, the norm defined by $\|x\|_W^*$ is a Lyapunov function for (4).

To estimate the domain of attraction of $x=0$ of (E), we pick $v(x) = \|x\|_W^*$ as a Lyapunov function and then we construct the gradient of $v(x)$, $\nabla v(x)$, normal to each flat determined by $v(x) = \|x\|_W^*$ (see Figure 1). The following discussion is phrased in terms of two-dimensional systems. This method must be modified when the dimension $n > 2$. We now fix ℓ_i points, $x^{i1}, \cdots, x^{i\ell_i}$, at uniform intervals on each line segment L_i forming the boundary of W^*, where ℓ_i is proportional to the length of L_i, $i=1$, $\cdots$, n_ℓ. Next we perform a direct search to determine a constant, $c_{min} = \min_i \{c_i\}$, such that

$$Dv(x) = \left[\nabla v(c_i x^{i_k}) \right]^T_{L_i} \cdot f(c_i x^{i_k}) < 0 \qquad (6)$$

$$k = 1, \cdots, \ell_i$$

$$i = 1, \cdots, n_\ell$$

where $[\nabla v(\cdot)]_{L_i}$ denotes the gradient vector evaluated on L_i. Note that on each L_i, the normal vector $[\nabla v(\cdot)]_{L_i}$ is a constant vector for all points on L_i. Thus for each L_i, $[\nabla v(\cdot)]_{L_i}$ needs to be computed only once. If there exists a constant c_{min}, $0 < c_{min} < \infty$, such that (6) holds, then $x=0$ of (E) is asymptotically stable for all points in W_D, where

$$W_D = c_{min} \cdot W^*,$$

i.e., W_D is a subset of the domain of attraction of $x=0$ of (E).
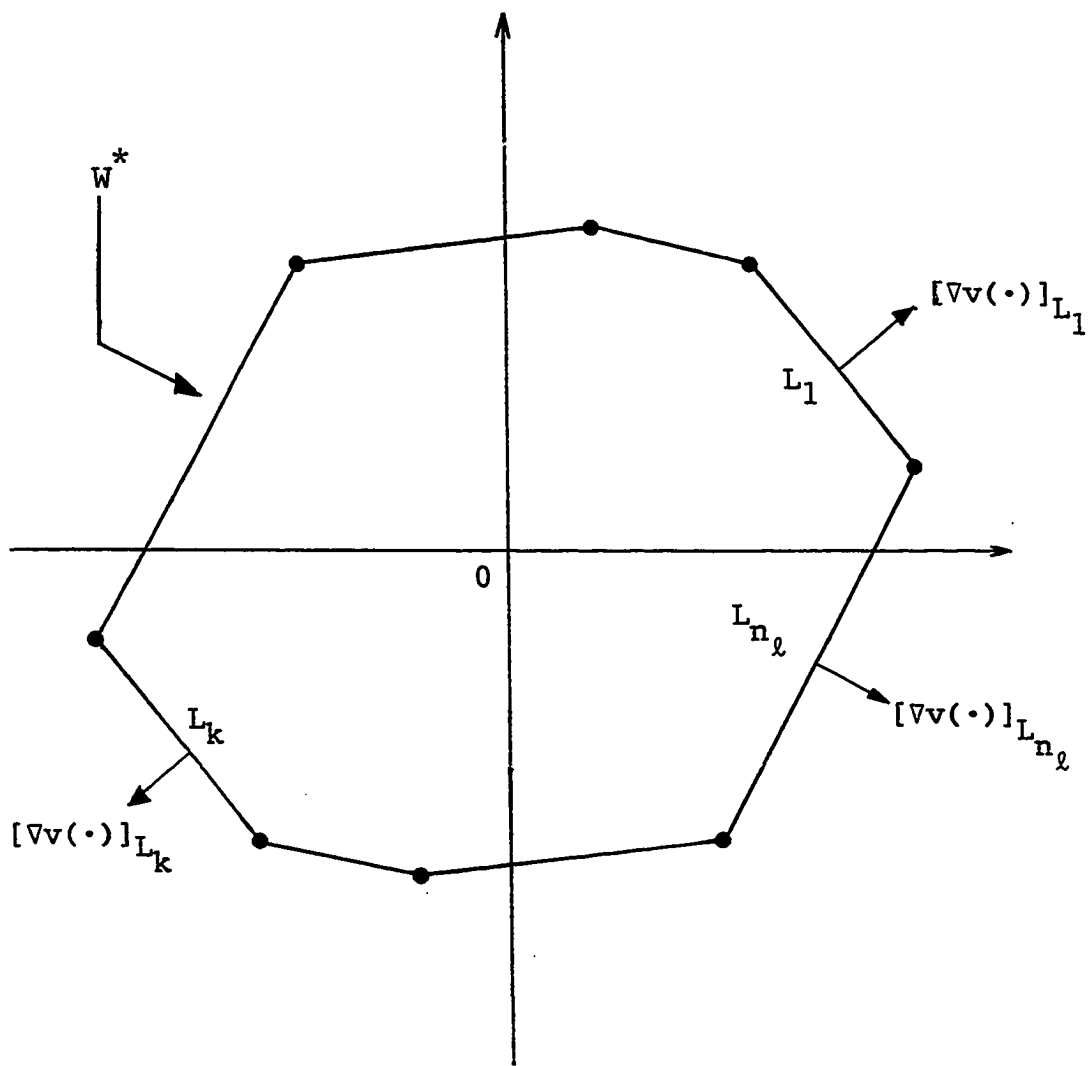
Figure 1. W* and its normal vectors

IV. INTERCONNECTED SYSTEMS

In this chapter, global stability results are presented for interconnected systems described by equations of the form

$$\dot{z}_i = F_i(z_i) + G_i(x), \qquad\qquad (\Sigma_i)$$

$i = 1, \cdots, \ell$, where $z_i \varepsilon R^{n_i}, F_i: R^{n_i} \rightarrow R^{n_i}$, and

$G_i: R^n \rightarrow R^{n_i}$, $x \varepsilon R^n$ with $x^T = (z_1^T, \cdots, z_\ell^T)$, and

$n = \sum\limits_{i=1}^{\ell} n_i$. Henceforth, it is assumed that $F_i(z_i) = 0$ if and only if $z_i = 0$, and $G_i(x) = 0$ if and only if x=0. Let $F(x)^T = [F_1(z_1)^T, \cdots, F_\ell(z_\ell)^T]$ and $G(x)^T = [G_1(x)^T, \cdots, G_\ell(x)^T]$. Then, (Σ_i) can be represented equivalently as

$$\dot{x} = F(x) + G(x) \triangleq H(x). \qquad\qquad (S)$$

Clearly, $F: R^n \rightarrow R^n$, $G: R^n \rightarrow R^n$, and $H: R^n \rightarrow R^n$. (S) is referred to as a composite system, or an interconnected system, or a large scale system with decomposition (Σ_i). Note that (Σ_i) may be veiwed as the interconnection of ℓ free

subsystems or isolated subsystems (S_i) described by the equations of the form

$$\dot{z}_i = F_i(z_i) \qquad\qquad (S_i)$$

for $i = 1, \cdots, \ell$.

For each $z_i \neq 0$, a matrix $M_i(z_i)$ is found so that $F_i(z_i) = M_i(z_i)z_i$. Then, (S_i) can be rewritten as

$$\dot{z}_i = M_i(z_i)z_i. \qquad\qquad (7)$$

Applying Euler's formula to (7), we obtain the system of difference equations given by

$$z_i(k+1) = [I_{n_i} + h_k M_i(z_i(k))]z_i(k) \qquad\qquad (S_i')$$

$i=1,\cdots,\ell$, where $k=0$ corresponds to t_0 in (7) and $h_k = t_{k+1} - t_k$, $k=0, 1, \cdots$. Let $S_i = \{M_i(z_i) \epsilon R^{n_i \times n_i}$: $M_i(z_i)$ is the chosen matrix for which $M_i(z_i)z_i = F_i(z_i)$ for $z_i \epsilon R^{n_i}\}$. According to the results of Brayton and Tong (see section III. B), if the set $A_i = \{I_{n_i} + h_i S_i\}$ is asymptotically stable for $h_i > 0$ sufficiently small, then the equilibrium $z_i = 0$ of the free subsystem (S_i) is globally asymptotically stable. Furthermore, only the set

of extremal matrices $E(A_i)$ needs to be considered in the analysis. Suppose that for each (S_i'), the set

$A_i = \{I_{n_i} + h_i S_i\}$ is stable for some $h_i > 0$ and that the set $\rho_i A_i$ is also stable for some $\rho_i > 1$. As shown in Section III.B, it is possible to study the stability properties of (S_i') in terms of the stability properties of the set A_i. Suppose that we have obtained for (S_i') the sets W_i^*, $i = 1, \cdots, \ell$, using the __constructive algorithm.__ Then a norm Lyapunov function $v_i(z_i) = \|z_i\|_i \overset{\Delta}{=} \|z_i\|_{W_i^*}$ for (S_i') is constructed using the set W_i^* (as discussed in Section III.B.). The set W_i^* determines the norm $\|\cdot\|_i$ by noting that the boundary of W_i^*, ∂W_i^*, consists of all points $z_i \in R^{n_i}$ such that $\|z_i\|_i = 1$, i.e.,

$$\partial W_i^* = \{z_i \in R^{n_i} : \|z_i\|_i = 1\}.$$

Since W_i^* is convex, for any $y_i \in R^{n_i}$ there is a $z_i \in \partial W_i^*$ such that $y_i = \alpha z_i$ for some $\alpha > 0$. For this α, $\|y_i\|_i = \alpha$. When $\alpha < 1$, y_i is in the interior of W_i^*, and when $\alpha > 1$, y_i is in the exterior of W_i^*. Also note that $v_i(z_i) = \|z_i\|_i$ is Lipschitz continuous with the Lipschitz constant equal to 1, i.e.,

$$|v_i(z_i') - v_i(z_i'')| \leqslant |z_i' - z_i''|_i \quad \text{for} \quad z_i', \; z_i'' \varepsilon R^{n_i}.$$

Lemma 1 [4], which follows, gives measures of the degree of asymptotic stability of the equilibrium $z_i=0$ of the free subsystems (S_i'). Also, an important result [9], Theorem 5, is presented in the following to determine the stability of the overall system (S).

Lemma 1. Suppose for the system

$$\dot{x} = F(x) = M(x)x \tag{E'}$$

there exist $h'>0$ and $\rho>1$ such that the set

$$\{\rho(I + h'M(x)): x\varepsilon R^n\} \tag{8}$$

is stable. Let W^* be the convex, balanced set determined by the method of Brayton and Tong for set (8). Let $v(x) \triangleq |x|_*$ be the corresponding norm Lyapunov function. Then, along the solutions of (E') we have

$$Dv_{(E')}(x) = \lim_{h \to 0^+} \sup \frac{v(x + h\,F(x)) - v(x)}{h}$$

$$\leqslant -\mu v(x)$$

where $\mu = (1 - \frac{1}{\rho})(\frac{1}{h},) > 0$.

Theorem 5. For the interconnected system

$$\dot{z}_i = F_i(z_i) + G_i(x), \qquad (\Sigma_i)$$

$$i = 1, \cdots, \ell$$

where $z_i \epsilon R^{n_i}$ and $x^T = (z_1^T, \cdots, z_\ell^T)$, the equilibrium $x=0$ of (Σ_i) is globally asymptotically stable if,

A-1)   each isolated subsystem (S_i): $\dot{z}_i = F_i(z_i)$

   satisfies Lemma 1, i.e., there exist $\mu_i > 0$ and a

   Lyapunov function $v_i(z_i) = \|z_i\|_i$ for (S_i) such

   that $Dv_{i(S_i)}(z_i) < - \mu_i \|z_i\|_i$ for all

   $z_i \epsilon R^{n_i}$, $i=1, \cdots, \ell$;

A-2)   for (Σ_i) there exist constants

   $g_{ij} > 0$, $i, j=1, \cdots, \ell$, such that

   $\|G_i(x)\|_i < \sum\limits_{j=1}^{\ell} g_{ij} \|z_j\|_j$ for all $x \epsilon R^n$;   and

A-3)   the successive principal minors of the

   $(\ell \times \ell)$ test matrix $D = [d_{ij}]$ are all positive,

   where $d_{ij} = \begin{cases} \mu_i - g_{ii}, & i = j \\ -g_{ij}, & i \neq j. \end{cases}$

In specific applications, usually more information concerning system structure is available than indicated in (Σ_i). In the following, it is assumed that the inter-connecting structure $G_i(x)$ in (Σ_i) can be written as

$$G_i(x) = \sum_{j=1}^{\ell} N_{ij}(x)z_j, \quad i=1,\cdots,\ell$$

for each $x \neq 0$, $x \varepsilon R^n$. Now, the particular case is considered when system (Σ_i) has a linear interconnecting structure of the form

$$G_i(x) = \sum_{j=1}^{\ell} A_{ij}z_j, \quad i=1,\cdots,\ell$$

where $A_{ij} \varepsilon R^{n_i \times n_j}$ are constant matrices (independent of x). In this case we have $g_{ij} = \|A\|_{ij}$. Given the norms $\|\cdot\|_i$ and $\|\cdot\|_j$ and the sets ∂W_i^* and ∂W_j^*, the norm $\|A\|_{ij}$ can be computed as follows: First, recall that

$$\|A_{ij}\|_{ij} = \sup_{\|z_j\|_j=1} \|A_{ij}z_j\|_i = \sup_{z_j \varepsilon \partial W_j^*} \|A_{ij}z_j\|_i.$$

Let $E(W_j^*)$ denote the extreme points of W_j^*. Since A_{ij} determines a linear map, the set

$$B \triangleq \{y_j = A_{ij}z_j : z_j \varepsilon E(W_j^*)\}$$

contains the extreme points of the image under A_{ij} of W_j^*.
Hence, it suffices to determine the norms of the elements
of B, i.e.,

$$\|A_{ij}\|_{ij} = \max\left\{\|A_{ij}z_j\|_i \colon z_j \in E(W_j^*)\right\}.$$

The results shown in this chapter make it possible to
utilize the constructive algorithm of Brayton and Tong
[1], [2] in the global asymptotic stability analysis of
interconnected dynamical systems, which may be of high
dimension. According to examples in [4], under certain
conditions, these results may offer advantages over
existing stability results for interconnected dynamical
systems [9]. In the next chapter, the results of the
present chapter will be used for the estimation of the
domain of attraction of equilibrium points of
interconnected systems which are locally asymptotically
stable.

V. DOMAIN OF ATTRACTION OF INTERCONNECTED SYSTEMS

In this chapter, an algorithm to estimate the domain of attraction (or stability region) of equilibrium points of interconnected systems is presented. If the free subsytems (S_i) are only locally asymptotically stable, then Theorem 5 in chapter IV will not be applicable in the stability analysis of the interconnected system (Σ_i). However, in this case, we can use the method of section III.C to obtain an estimate for the domain of attraction for each (S_i), i=1, $\cdots$,ℓ, and we can then attempt to modify Theorem 5 to obtain an estimate for the domain of attractin of x=0 for the interconnected system (Σ_i). In the following, a method for accomplishing this is presented.

As we discussed in Chapter IV, large scale systems are considered which are described by

$$\dot{z}_i = F_i(z_i) + G_i(x) \qquad\qquad (\Sigma_i)$$

for i=1,$\cdots$,ℓ, where $z_i \varepsilon R^{n_i}$,$x^T = (z_1^T, \cdots, z_\ell^T)$ and

$n = \sum\limits_{i=1}^{\ell} n_i$. Let $F(x)^T = (F_1(z_1)^T, \cdots, F_\ell(z_\ell)^T)$

and $G(x)^T = (G_1(x)^T, \cdots, G_\ell(x)^T)$. Then, (Σ_i) can be rewritten as

$$\dot{x} = F(x) + G(x) \triangleq H(x) \tag{S}$$

where H: $B(r) \to R^n$, for some $r > 0$, where

$B(r) = \{x \epsilon R^n: |x| < r\}$. Here, the isolated subsystems are described by

$$\dot{z}_i = F_i(z_i) \tag{S_i}$$

for i=1, $\cdots, \ell$, where $F_i: B_i(r_i) \to R^{n_i}$ for some $r_i > 0$ and where $B_i(r_i) = \{z_i \epsilon R^{n_i}: |z_i| < r_i\}$. For each isolated subsystem (S_i), we obtain a Lyapunov function $v_i(z_i) = \|z_i\|_i$ via the <u>constructive algorithm,</u> using the method of Section III.C., such that $Dv_{i(S_i)}(z_i)$ is negative definite in a neighborhood given by

$$C_i = \{z_i \epsilon R^{n_i}: v_i(z_i) = \|z_i\|_i < V_i^o\}. \tag{9}$$

As noted in Section III.C, the set C_i is contained in the domain of attraction of the equilibrium $z_i = 0$ for (S_i). To obtain the best estimate, we choose V_i^o in (9) as large as possible.

To obtain an estimate for the domain of attraction of
x=0 for the overall system (S), we use the Lyapunov
function v(x) defined on the domain B(r), for some
r>0, where

$$v(x) = \sum_{i=1}^{\ell} \alpha_i v_i(z_i) = \sum_{i=1}^{\ell} \alpha_i \|z_i\|_i. \tag{10}$$

Here, the α_i's are weighting factors and $\alpha_i > 0$,
i=1, $\cdots$,ℓ. Recall that an estimate of the domain of
attraction of x=0 for (S) is given by [8]

$$\Omega = \{x \varepsilon R^n : v(x) < V^o, Dv_{(S)}(x) < 0\}.$$

As pointed out by Weissenberger [12], the set

$$C_v = \{x \varepsilon R^n : v(x) = \sum_{i=1}^{\ell} \alpha_i v_i(z_i) < V^o\}$$

is a subset of the domain of attraction of x=0 for (S),
where $V^o = \min_{1 \le i \le \ell} \{\alpha_i V_i^o\}$, if the constraints given by
hypotheses (A-1) - (A-3) in Theorem 5 are satisfied over
C_v.

In order to obtain the best estimate by this method,
we must choose the α_i in an optimal fashion. Several
techniques have been used to accomplish this. We follow

here [13] and [14], where two methods are suggested, both
of which are amenable to linear programming techniques.
In the first of these methods, we make use of the
following dominance conditions for M-matrices: An
($\ell \times \ell$) matrix $D = [d_{ij}]$ is an M-matrix if and only if there
exist positive constants λ_j, j=1, $\cdots$, ℓ, such that

$$\sum_{j=1}^{\ell} \lambda_j d_{ij} > 0, \quad i=1, \cdots, \ell, \text{ where } d_{ij} < 0 \text{ for } i \neq j.$$

<u>Method 1.</u> Minimize the trace of test matrix D in
Theorem 5, i.e.,

$$\text{minimize } z = \sum_{i=1}^{\ell} \alpha_i |\mu_i - g_{ii}|, \tag{11}$$

subject to the constraints

$$\alpha_i |\mu_i - g_{ii}| - \sum_{\substack{j=1 \\ i \neq j}}^{\ell} \alpha_j g_{ij} > 0, \tag{12}$$

$$i=1, \cdots, \ell, \text{ and } \sum_{i=1}^{\ell} \alpha_i = 1.$$

<u>Method 2.</u> Maximize the weighted sum of the
"hypervolumes" for the estimates C_i of (9), i.e.,

$$\text{maximize} \quad z = \sum_{i=1}^{\ell} \alpha_i V_i^O \qquad\qquad (13)$$

subject to the constraints (12), where the V_i^O are defined in (9).

We denote the α_i's obtained either by Method 1 or Method 2 by α_i^*'s. Thus, the estimate of the domain of attraction for $x=0$ of system (S) will be given by

$$\tilde{D} = \left\{ x \varepsilon R^n : v(x) = \sum_{i=1}^{\ell} \alpha_i^* \|z_i\|_i < c \right\}$$

where $c = \min_{i} \{\alpha_i^* V_i^O\}$. We note that the optimization procedure in Method 1 has the effect of reducing the "eccentricity" of the estimate $\tilde{D}$.

In summary, the above procedure makes it possible to apply the constructive algorithm in estimating the domain of attractin of $x=0$ for an interconnected system (Σ_i) which need not be of low dimension.

VI. APPLICATION TO POWER SYSTEMS

The results shown in the previous chapter will now be used to find an estimate of the stability region of a multimachine power system and to find the critical clearing time of a faulted power system.

Any physical dynamical system that is designed to perform certain preassigned tasks in a steady state mode must, in addition to performing these functions in a satisfactory manner, be stable at all times relative to sudden disturbances with an adequate margin of safety. At the design stage, many contingencies are taken into consideration. But, in the subsequent operation and augmentation of the network, new conditions arise which can not be foreseen. Hence, an entirely different pattern of system behavior can be expected under actual operating conditions. This is particularly true regarding the capability of a power system to maintain synchronism or stabilty due to sudden unforeseen disturbances, such as loss of a major transmission line or load. The tools suitable for off-line studies, such as simulation, may not be suitable for on-line application, since a large number of contingencies have to be simulated in a short time. A technique which offers promise for this purpose is

Lyapunov's direct method. The appeal of this method lies in its ability to directly compute the critical clearing time of circuit breakers for various faults and thus directly assess the degree of stability for a given configuration and operating state.

Lyapunov's method was first proposed as a solution to the power system stability problem by Gless [15] and El-Abiad and Nagappan [5] in 1966. For the detailed history of the applications of the second method of Lyapunov to power systems, refer to [13].

The ideas presented in this chapter were motivated by [14]. A short summary of this chapter is as follows: An n-machine power system with uniform damping charac-teristics and with transfer conductance is decomposed into (n-1) subsystems. The norm Lyapunov functions and the stability regions of the isolated subsystems are determined using the constructive algorithm discussed in sections B and C of Chapter III. A Lyapunov function of the overall system is constructed using a weighted sum of the Lyapunov functions of the isolated subsystems. A test matrix is formed using the constraints of the stability of the isolated subsystems and the interconnection characteristics. If the test matrix is an M-matrix, then the overall system is asymptotically stable in some domain.

The remainder of this chapter will be devoted to:

i)   finding the stability region of a 4-machine
     power system and

ii) estimating the critical clearing time when the
     power system is faulted during the steady-state
     operation.

A.   Stability Region of a Power System

Consider a power system composed of n synchronous machines interconnected through a network. The following assumptions [14] are made:

1.   A synchronous machine will be represented by a
     constant voltage in series with its transient
     reactance during the post-fault period.

2.   The mechanical input power is constant during
     the transient period.

3.   The loads are approximated by constant
     admittances.

4. The damping coefficients are assumed constant. In order to keep the problem relatively simple, uniform damping is also assumed, i.e.,

$$r = \frac{D_1}{M_1} = \frac{D_2}{M_2} = \cdots = \frac{D_n}{M_n}.$$

With the above assumptions, the system model being considered in this investigation has a dynamic equation set:

$$M_i \dot{w}_i + D_i w_i = P_{mi} - P_{ei} \qquad (14)$$

$$\dot{\delta}_i = w_i \qquad (15)$$

$$\text{where } P_{mi} = \sum_{j=1}^{n} E_i E_j Y_{ij} \cos (\delta_{ij}^{\;O} - \theta_{ij}), \text{ and} \qquad (16)$$

$$P_{ei} = \sum_{j=1}^{n} E_i E_j Y_{ij} \cos (\delta_{ij} - \theta_{ij}) \qquad (17)$$

for i=1, $\cdots$, n.

For the i-th subsystem (or i-th machine),

w_i, δ_i = absolute rotor speed and angle, respectively;

M_i, D_i = moment of inertia and damping coefficient, respectively;

P_{mi} = mechanical input power (constant);

P_{ei} = electrical output power;

E_i = internal voltage (constant);

Y_{ij}, θ_{ij} = magnitude and angle in the reduced bus admittance matrix, respectively;

$\delta_{ij} = \delta_i - \delta_j$, $\delta_{ij}^{o} = \delta_i^{o} - \delta_j^{o}$; and

δ_i^{o} = absolute rotor angle at post-fault equilibrium state.

Let the angle and angular velocity differences with respect to a reference machine n be

$$\delta_{in} = \delta_i - \delta_n \text{ an } w_{in} = w_i - w_n,\text{ respectively.}$$

Let $z_i = (x_{1i}, x_{2i})^T = (\delta_{in} - \delta_{in}^{o}, w_{in})^T$. Then, from equations (14) - (17) we obtain

$$\dot{z}_i = A_i z_i - b\, f_i(y_i) + G_i(x) \qquad (\Sigma_i)$$

$i=1, \cdots, n-1$, where

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & -r \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad y_i = [1 \quad 0]z_i,$$

$$G_i(x) = \begin{bmatrix} 0 \\ g_i(x) \end{bmatrix}, \quad x^T = (z_1^T, \cdots, z_{n-1}^T),$$

$$f_i(y_i) = \mu_{1i}[\sin(y_i + \delta_{in}^O) - \sin \delta_{in}^O]$$

$$+ \mu_{2i}[\cos(y_i + \delta_{in}^O) - \cos \delta_{in}^O] , \tag{18}$$

$$\mu_{1i} = (\frac{1}{M_i} + \frac{1}{M_n}) E_i E_n Y_{in} \sin\theta_{in} ,$$

$$\mu_{2i} = (\frac{1}{M_i} - \frac{1}{M_n}) E_i E_n Y_{in} \cos \theta_{in};$$

and

$$g_i(x) = \frac{1}{M_n} \sum_{\substack{j=1 \\ j \neq i}}^{n-1} E_n E_j Y_{nj} [\cos(-x_{1j} + \delta_{nj}^O - \theta_{nj})$$

$$-\cos\ (\delta_{nj}^{o}\ -\ \theta_{nj})]$$

$$-\ \frac{1}{M}\ \sum_{\substack{i \ \ j=1 \\ j\neq i}}^{n-1}\ E_i E_j Y_{ij}[\cos\ (x_{1i}\ -\ x_{1j}\ +\ \delta_{ij}^{o}\ -\ \theta_{ij})$$

$$-\ \cos\ (\delta_{ij}^{o}\ -\ \theta_{ij})]. \hspace{3cm} (19)$$

For a 4-machine power system, there are 3 subsystems, with machine 4 used as a reference. Let $x_1 = x_{11}$, $x_2 = x_{21}$,

$x_3 = x_{12}$, $x_4 = x_{22}$, $x_5 = x_{13}$, and $x_6 = x_{23}$. Let

$z_1^T = (x_1,\ x_2)$, $z_2^T = (x_3,\ x_4)$, and $z_3^T = (x_5,\ x_6)$. Let (S_i) be the isolated subsystem of (Σ_i):

$$\dot{z}_i\ =\ A_i z_i\ -\ bf_i(y_i)\ \triangleq\ F_i(z_i) \hspace{2cm} (S_i)$$

where $b^T = (0\ 1)$ and $y_i = (1\ 0)z_i$. Then, (Σ_i) above can be rewritten as

$$\dot{z}_i\ =\ F_i(z_i)\ +\ G_i(x) \hspace{3cm} (\Sigma_i)$$

for $i = 1,\ 2,\ 3$. Let $F(x)^T = (F_1(z_1),\ F_2(z_2),\ F_3(z_3))$ and

$G(x)^T = (G_1(x), G_2(x), G_3(x))$. Then, $z_i=0$ is an equilibrium point of (S_i) and $x=0$ is an equilibrium point of the overall system (S) described by

$$\dot{x} = F(x) + G(x). \qquad \qquad (S)$$

Table I shows the machine constants of the 4-machine power system considered. The values of the parameters are taken in per unit. Table II shows the reduced bus admittance matrix for the post-fault system, and Table III shows the internal voltage for the post-fault system.

Table I. Machine Constants

Generator number	M_i in P.U.	D_i in P.U.	$r = \dfrac{D_i}{M_i}$	P_{mi} in P.U.
1	0.008	0.112	14	0.100
2	0.016	0.224	14	0.300
3	0.0106	0.1484	14	0.200
4	0.5302	7.4228	14	0.335

Table II. Reduced Bus Admittance Matrix $Y_{ij} \angle \theta_{ij}$

for Post-Fault System

i\j	1	2	3	4
1	0.88 $\angle$ -88.1°	0.124 $\angle$ 82.1°	0.065 $\angle$ 82.4°	0.658 $\angle$ 91.1°
2	0.124 $\angle$ 82.1°	0.873 $\angle$ -83.2°	0.064 $\angle$ 88.2°	0.655 $\angle$ 96.8°
3	0.065 $\angle$ 82.4°	0.064 $\angle$ 88.2°	1.014 $\angle$ -75.5°	0.754 $\angle$ 99.0°
4	0.658 $\angle$ 91.1°	0.655 $\angle$ 96.8°	0.754 $\angle$ 99.0°	2.447 $\angle$ -69.7°

Table III. Internal Voltage for Post-Fault System

Generator number	$E_i \angle \delta_i$
1	1.057 $\angle$ 5.69°
2	1.155 $\angle$ 14.39°
3	1.095 $\angle$ 2.27°
4	1.000 $\angle$ 0.08°

Note that $P_{mi} = \sum_{j=1}^{n} E_i E_j Y_{ij} \cos (\delta_{ij}^{o} - \theta_{ij})$ is the

constant mechanical input, where $\delta_{ij}^{o} = \delta_i^{o} - \delta_j^{o}$,

$i,j=1,\cdots,n$.

Using the values of the parameters in Tables I-III, the equations for the isolated subsystems (S_i) are obtained as follows:

$$(S_1): \quad \dot{z}_1 = F_1(z_1): \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -14x_2 - f_1(x_1) \end{cases}$$

where, from (18),

$$f_1(x_1) = 88.23 [\sin (x_1 + 0.0979) - 0.09776]$$

$$- 1.6438 [\cos (x_1 + 0.0979) - 0.9952];$$

$$(S_2): \quad \dot{z}_2 = F_2(z_2): \begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = -14x_4 - f_2(x_3) \end{cases}$$

where $f_2(x_3) = 48.367 [\sin (x_3 + 0.2498) - 0.2472]$

$$- 5.4295 [\cos (x_3 + 0.2498) - 0.96897];$$

and

$$(S_3): \quad \dot{z}_3 = F_3(z_3): \begin{cases} \dot{x}_5 = x_6 \\ \dot{x}_6 = -14x_6 - f_3(x_5) \end{cases}$$

where $f_3(x_5) = 78.4687$ [sin $(x_5 + 0.03822) - 0.0382$]

$$- 11.941 \text{ [cos } (x_5 + 0.03822) - 0.99927].$$

The generation of the final convex sets W_i^*, i=1,2,3, makes use of the constructive algorithm discussed in section III.B. First, Jacobian matrices $J_i = \dfrac{\partial F_i}{\partial z_i}\Big|_{z_i=0}$ are computed as below:

$$J_1 = \begin{bmatrix} 0 & 1 \\ -88 & -14 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 1 \\ -48.21 & -14 \end{bmatrix}, \text{ and}$$

$$J_3 = \begin{bmatrix} 0 & 1 \\ -78.87 & -14 \end{bmatrix}.$$

Next $h_i > 0$ is chosen such that $|\lambda(M_i)| < 1$, i = 1,2,3, where $M_i = I + h_i J_i$. The values of the h_i's are: $h_1 = 0.02$, $h_2 = 0.1$, and $h_3 = 0.09$. The corresponding matrices M_i are:

$$M_1 = \begin{bmatrix} 1 & 0.02 \\ -1.76 & 0.72 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0.1 \\ -4.821 & -0.4 \end{bmatrix}, \text{ and}$$

$$M_3 = \begin{bmatrix} 1 & 0.09 \\ -7.0983 & -0.26 \end{bmatrix}.$$

Using Brayton and Tong's algorithm of section III.B, the final convex sets W_i^*, i =1,2,3 are generated, picking an initial convex and balanced set W_{oi} containing the origin. The Lyapunov functions $v_i(z_i) = \|z_i\|_i$, i=1,2,3 are now obtained from the plots of the boundaries of W_i^* generated by the computer for each (S_i), i=1,2,3. Next, using the algorithm discussed in section III.C, the stability regions C_i of the isolated subsystems (S_i) are estimated as below:

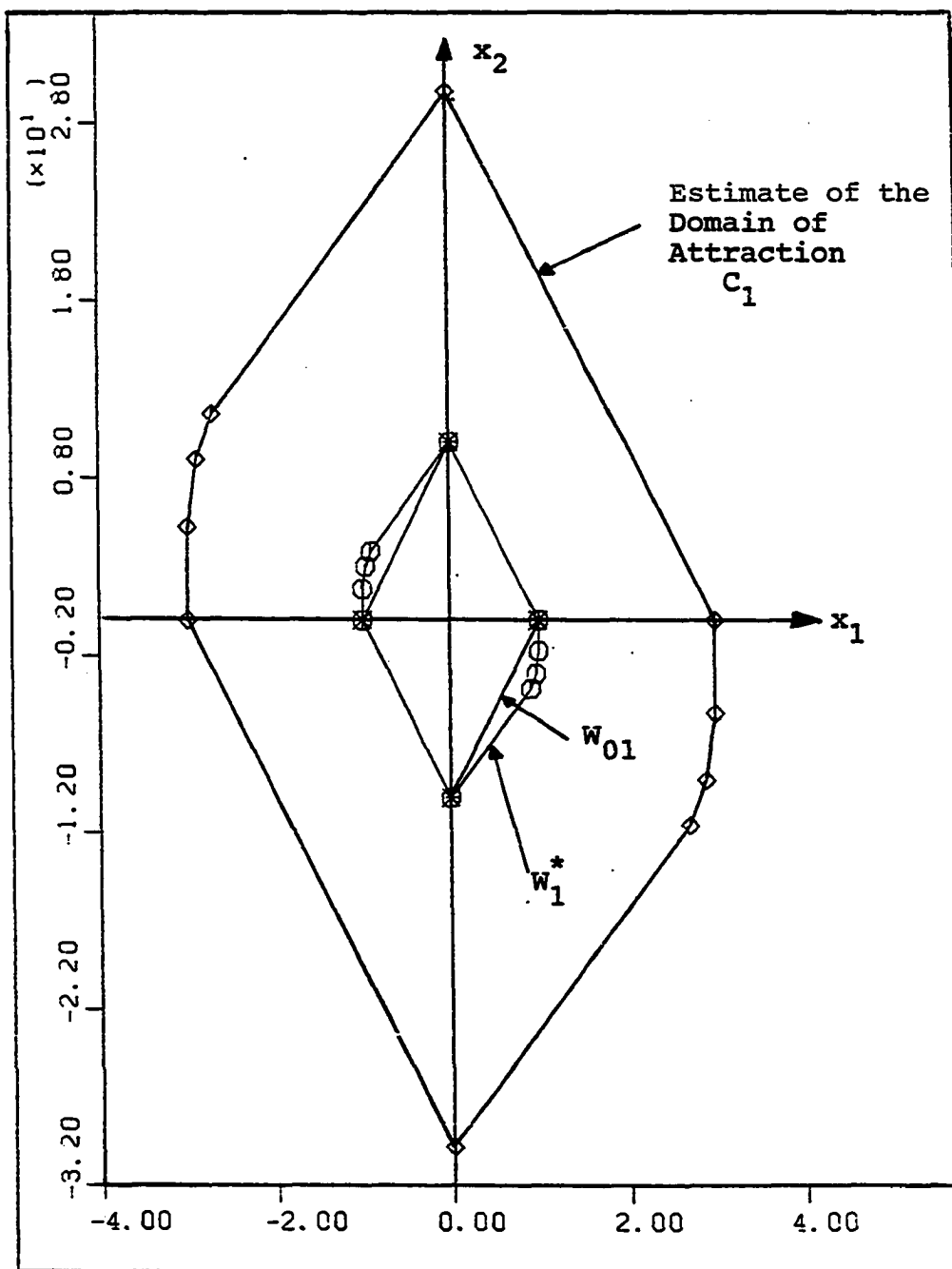$$C_i = \{z_i \varepsilon R^2 : v_i(z_i) = \|z_i\|_i < V_i^o\},$$

where $\quad V_1^o = 2.9830,$

$$V_2^o = 2.8656, \text{ and}$$

$$V_3^o = 2.9173.$$

Note that V_i^o is dependent on the choice of the value of h_i as well as the shape of the initial convex set W_{oi}, and also that W_i^* is dependent on both h_i and W_{oi}. The graphs of W_i^* and the stability regions of each (S_i) are shown in Figures 2, 3 and 4.

Recall that a set of matrices A is asymptotically
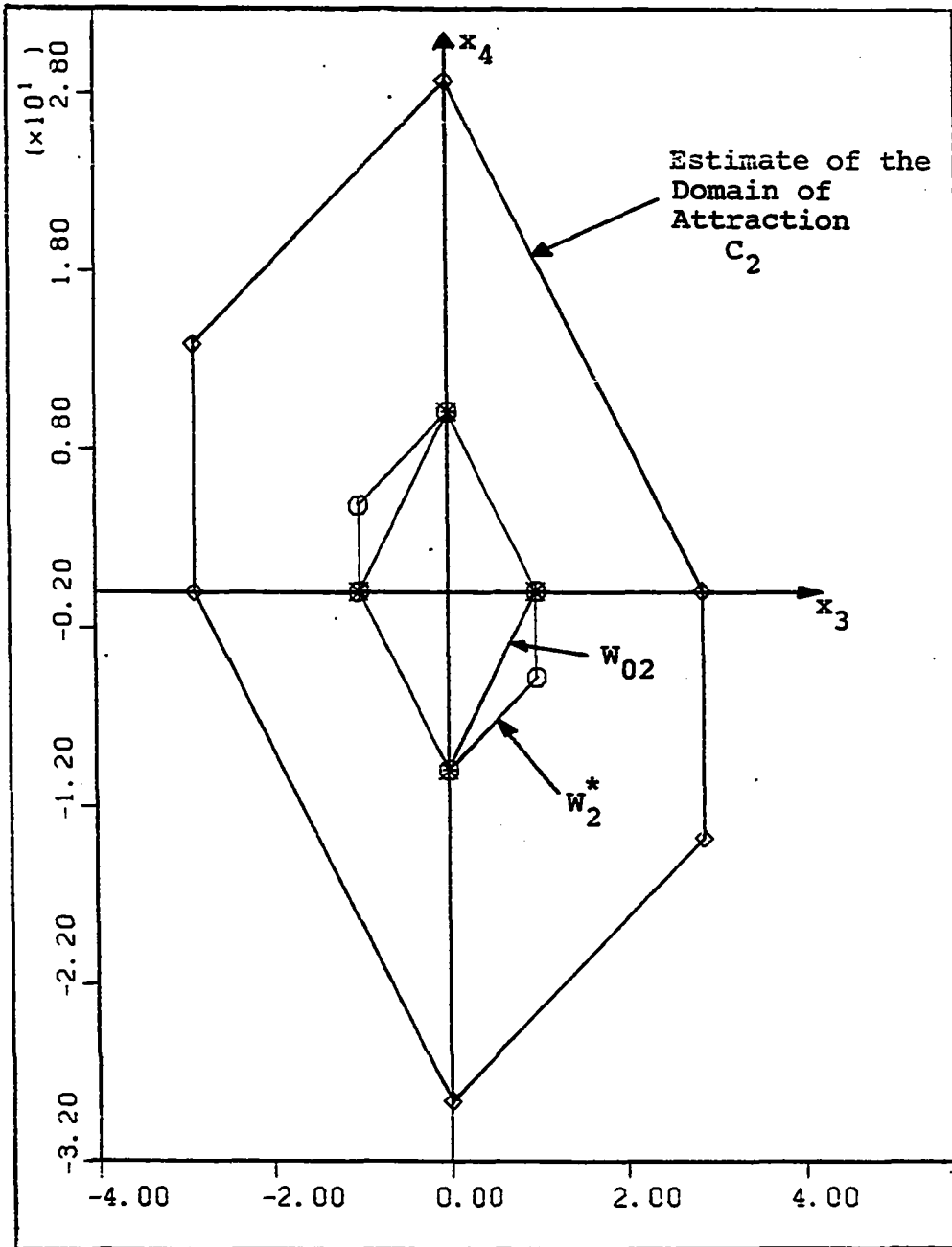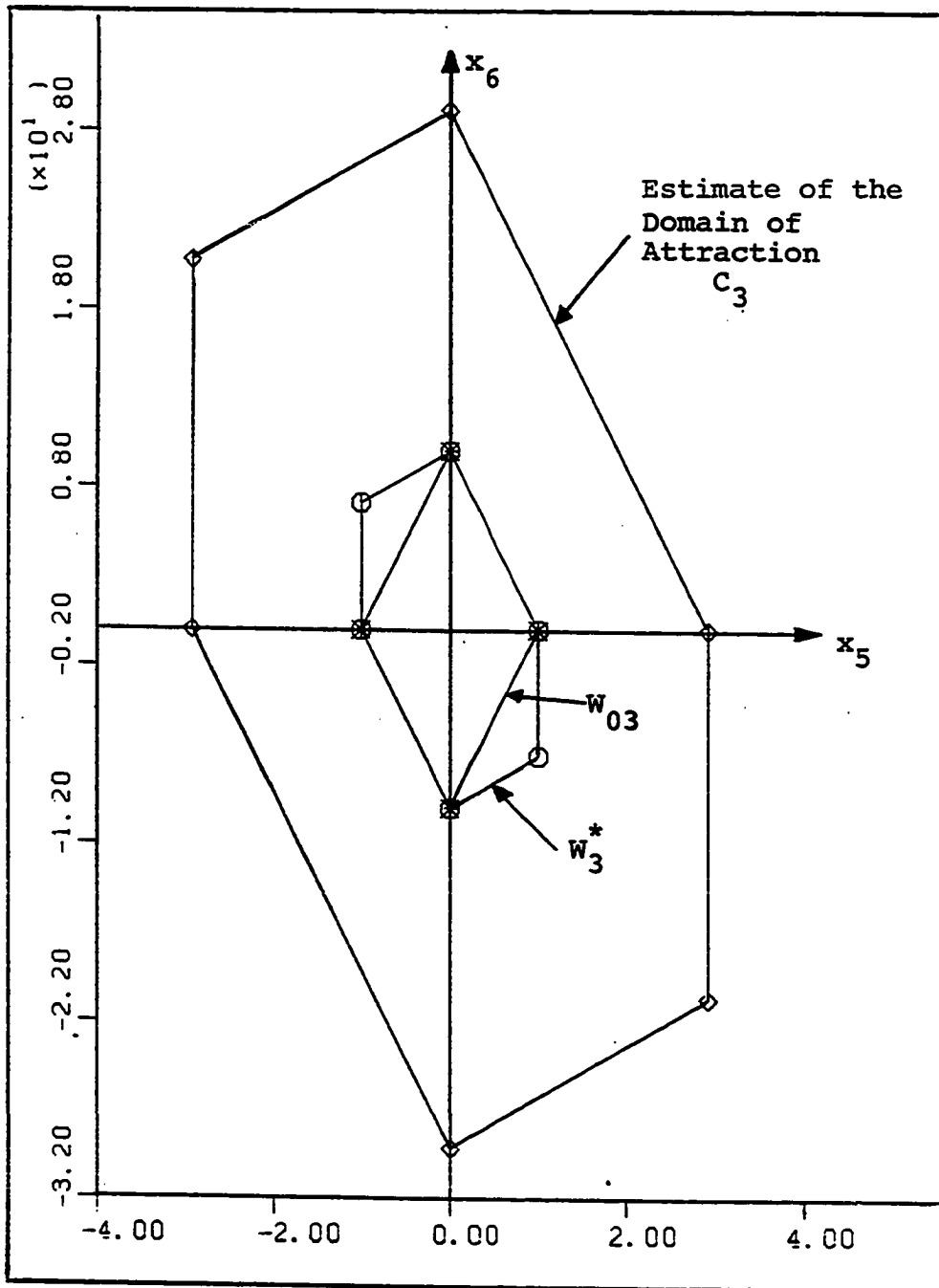
Figure 2. W_1^* and C_1

Figure 3. W_2^* and C_2

Figure 4. W_3^* and C_3

stable if there exists a number $\rho > 1$ such that ρA is stable. The values of ρ_i for (S_i) are computed as:

$$\rho_1 = 1.14, \quad \rho_2 = 2.57, \quad \text{and} \quad \rho_3 = 1.62.$$

Using Lemma 1 of chapter IV, the stability measures μ_i of the isolated subsystems (S_i) are computed as:

$$\mu_1 = 6.1404, \quad \mu_2 = 6.1089, \quad \text{and} \quad \mu_3 = 4.2524$$

where $\mu_i = (1 - \frac{1}{\rho_i})(\frac{1}{h_i})$.

For the stability analysis of the overall power system (S), Theorem 5 of chapter IV will be used. Recall that the test matrix $D = [d_{ij}]$ of this theorem is:

$$d_{ij} = \begin{cases} \mu_i - g_{ii}, & i = j \\ -g_{ij}, & i \neq j, \end{cases}$$

where g_{ij}, i, $j = 1, 2, 3$ are computed from the inter-connecting structures $G_i(x)^T = (0 \ g_i(x))$, using the norms $\|\cdot\|_i$ and $\|\cdot\|_j$ which are induced by W_i^*, $i = 1, 2, 3$. To estimate the function $G_i(x)$ by the Lyapunov function $v_i(z_i) = \|z_i\|_i$ to obtain g_{ij}, the following two inequalities will be used:

i) $a[\cos(y+\theta) - \cos \theta] \le |a| \cdot |\sin \theta| \cdot |y|$

for all a, y, and θ in R, and

ii) $|y_1 \pm y_2| \le |y_1| + |y_2|$ for all y_1 and y_2 in R.

It is seen from $G_i(x)^T = (0 \; g_i(x))$, in the expression Σ_i, that as $|g_i(x)|$ increases, $\|G_i(x)\|_{W_i}^*$ increases. Using the data from Tables I-III and equation (19), we obtain

$$g_1(x) = a_{11} (\cos(-x_3 + \theta_1) - \cos \theta_1)$$

$$+ a_{12} (\cos(-x_5 + \theta_2) - \cos \theta_2)$$

$$+ a_{13} (\cos(x_1 - x_3 + \theta_3) - \cos \theta_3)$$

$$+ a_{14} (\cos(x_1 - x_5 + \theta_4) - \cos \theta_4),$$

where

$a_{11} = 1.427$, $a_{12} = 1.557$, $a_{13} = -19.923$, $a_{14} = -9.404$
and

$\theta_1 = -111.11^\circ$, $\theta_2 = -101.19^\circ$, $\theta_3 = -90.8^\circ$, $\theta_4 = -78.98^\circ$.

Next, using the above inequalities, we obtain

$$|g_1(x)| \le |a_{11}| \cdot |\sin \theta_1| \cdot |x_3| + |a_{12}| \cdot |\sin \theta_2| \cdot |x_5|$$

$$+ |a_{13}| \cdot |\sin \theta_3| \cdot (|x_1| + |x_3|)$$

$$+ |a_{14}| \cdot |\sin \theta_4| \cdot (|x_1| + |x_5|)$$

$$\triangleq b_{11}|x_1| + b_{12}|x_3| + b_{13}|x_5|,$$

where $b_{11} = 28.152$, $b_{12} = 20.252$, $b_{13} = 10.759$. Also,

$$G_1(x) = \begin{bmatrix} 0 \\ g_1(x) \end{bmatrix} \le \begin{bmatrix} 0 & 0 \\ b_{11} & 0 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b_{12} & 0 \end{bmatrix} \begin{bmatrix} |x_3| \\ |x_4| \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ b_{13} & 0 \end{bmatrix} \begin{bmatrix} |x_5| \\ |x_6| \end{bmatrix}$$

$$\triangleq \sum_{j=1}^{3} B_{1j} z_j \text{ for } x_i \ge 0, \text{ where } B_{1j} = \begin{bmatrix} 0 & 0 \\ b_{1j} & 0 \end{bmatrix}.$$

For $G_2(x)^T = (0 \; g_2(x))$ from (Σ_2), we obtain

$$g_2(x) = 1.312(\cos(-x_1 + \theta_1) - \cos \theta_1)$$

$$+ 1.557 \ (\cos(-x_5 + \theta_2) - \cos \theta_2)$$

$$-9.461 \ (\cos(x_3 - x_1 + \theta_3) - \cos \theta_3)$$

$$-5.059 \ (\cos(x_3 - x_5 + \theta_4) - \cos \theta_4)$$

where

$$\theta_1 = -96.71^0, \ \theta_2 = -101.19^0, \ \theta_3 = -73.4^0, \ \theta_4 = -76.08^0.$$

In a similar way, we obtain the estimates

$$|g_2(x)| < b_{21}|x_1| + b_{22}|x_3| + b_{23}|x_5|,$$

where $b_{21} = 10.370, \ b_{22} = 13.977, \ b_{23} = 6.437$

and

$$G_2(x) = \begin{bmatrix} 0 \\ g_2(x) \end{bmatrix} < \begin{bmatrix} 0 & 0 \\ b_{21} & 0 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b_{22} & 0 \end{bmatrix} \begin{bmatrix} |x_3| \\ |x_4| \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ b_{23} & 0 \end{bmatrix} \begin{bmatrix} |x_5| \\ |x_6| \end{bmatrix}$$

$$= \sum_{j=1}^{3} B_{2j} z_j \text{ for } x_i > 0, \text{ where } B_{2j} = \begin{bmatrix} 0 & 0 \\ b_{2j} & 0 \end{bmatrix}.$$

For $G_3(x)^T = (0 \ g_3(x))$ from (Σ_3), we obtain

$$g_3(x) = 1.312 \ (\cos(-x_1 + \theta_1) - \cos \theta_1)$$

$$+ 1.427 \ (\cos(-x_3 + \theta_2) - \cos \theta_2)$$

$$- 7.097 \ (\cos(x_5 - x_1 + \theta_3) - \cos \theta_3)$$

$$- 7.636 \ (\cos(x_5 - x_3 + \theta_4) - \cos \theta_4)$$

where

$$\theta_1 = -96.71^\circ, \ \theta_2 = -111.11^\circ, \ \theta_3 = -85.82^\circ, \ \theta_4 = -100.32^\circ.$$

Thus, we obtain

$$|g_3(x)| \leq b_{31}|x_1| + b_{32}|x_3| + b_{33}|x_5|,$$

where $b_{31} = 8.381$, $b_{32} = 8.843$, $b_{33} = 14.590$, and

$$G_3(x) = \begin{bmatrix} 0 \\ g_3(x) \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ b_{31} & 0 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b_{32} & 0 \end{bmatrix} \begin{bmatrix} |x_3| \\ |x_4| \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ b_{33} & 0 \end{bmatrix} \begin{bmatrix} |x_5| \\ |x_6| \end{bmatrix}$$

$$= \sum_{j=1}^{3} B_{3j} z_j \text{ for } x_i > 0, \text{ where } B_{3j} = \begin{bmatrix} 0 & 0 \\ b_{3j} & 0 \end{bmatrix}.$$

Now, recall that if $G_i(x)$ is of the form

$$G_i(x) = \sum_{j=1}^{\ell} A_{ij} z_j, \quad i=1, \cdots, \ell,$$

where $A_{ij} \epsilon R^{n_i \times n_j}$ are constant matrices (independent of x), then

$$\|A_{ij}\|_{ij} = \max \left\{ \|A_{ij} z_j\|_{W_i^*} : z_j \epsilon E(W_j^*) \right\}.$$

Therefore, $g_{ij} = \|B_{ij}\|_{ij}$, $i,j = 1,2,3$, and these are computed from W_i^* and W_j^* as follows:

$$g_{11} = 2.8152 \qquad g_{12} = 2.0252 \qquad g_{13} = 1.0759$$

$$g_{21} = 1.037 \qquad g_{22} = 1.3977 \qquad g_{23} = 0.6437$$

$$g_{31} = 0.8381 \qquad g_{32} = 0.8843 \qquad g_{33} = 1.4590.$$

Using machine 4 as a reference and letting

$\alpha_i = 1$, i=1, 2,3, the following test matrix $D = [d_{ij}]$ is

formed:

$$
D = \begin{bmatrix} 3.3252 & -2.0252 & -1.0759 \\ -1.0370 & 4.7112 & -0.6437 \\ -0.8381 & -0.8843 & 2.7934 \end{bmatrix}.
$$

The above is an M-matrix, since the diagonal dominance conditions are satisfied.

Let $v(x) = \sum\limits_{i=1}^{3} \alpha_i v_i(z_i)$ be the Lyapunov function of the overall system (S). Then, since $D = [d_{ij}]$ is an M-matrix, the time derivative along the solution of (S) is negative definite, i.e.,

$$
Dv_{(S)}(x) < -\alpha^T Dw,
$$

where

$$
\alpha^T = (\alpha_1 \alpha_2 \alpha_3), \ \alpha_i > 0 \text{ and } w^T = (v_1(z_1), v_2(z_2), v_3(z_3)).
$$

Therefore, by Theorem 5, the equilibrium x=0 of the overall system (S) is asymptotically stable in some region.

To obtain an estimate of the stability region as large as possible, Criterion I or II can be employed using linear programming. By Criterion I, we obtain

$$\alpha_1^* = 0.2747, \quad \alpha_2^* = 0.1404, \quad \alpha_3^* = 0.5849, \text{ and}$$

$$V^o = \min_{i}\{\alpha_i^* V_i^o\} = 0.4023.$$

Thus, the estimated stability region of the equilibrium x=0 of the overall system (S) is the set

$$\left\{ x \epsilon R^{2(n-1)} : \sum_{i=1}^{n-1} \alpha_i^* \|z_i\|_i < 0.4023, \; n=4 \right\}.$$

By Criterion II, we obtain

$$\alpha_1^* = 0.6026, \quad \alpha_2^* = 0.1645, \quad \alpha_3^* = 0.2329, \text{ and}$$

$$V^o = \min_{i}\{\alpha_i^* V_i^o\} = 0.47139.$$

An estimate of the stability region of x=0 of (S) is the set

$$\left\{ x \epsilon R^{2(n-1)} : \sum_{i=1}^{n-1} \alpha_i^* \|z_i\|_i < 0.47139, \; n=4 \right\}.$$

The algorithm of Section III.C was coded in Fortran
WATFIV on the ITEL AS/6 system, and the ZX3LP subroutine
available in the IMSL subroutine package was used to solve
the linear programming problem for α_i^*, i=1,2,3.

B. Estimation of Critical Clearing Time

A figure of merit which has been used extensively in
power system transient stability studies is the so-called
critical clearing time t_c. By this we mean the largest
point in time up to which the disturbance in a power
system may persist, without the power system losing
synchronism. In practice, a value for t_c is usually
obtained by numerical simulation of differential
equations. However, it turns out that the Lyapunov
results obtained in the previous section can also be used
to estimate t_c. This will enable us to determine how good
(or how conservative) the above results are, at least as
far as the present power system is concerned.

Transient stability studies of power systems require
consideration of

      i)   the pre-fault system,

     ii)   the faulted system, and

    iii)   the post-fault system.

The pre-fault system, which is assumed to be operating in a steady state with mechanical power inputs equaling electrical power outputs plus transmission losses (neglecting losses in machines), is completely described by a set of nonlinear algebraic equations known as the load flow equations. The rotor angles δ_i^o of the generators are measured with respect to a synchronously rotating reference frame. When a fault occurs at time $t=t_o$, the balance between the mechanical input power and elecrical output power at each of the generators is upset, and as a consequence, some of the generators accelerate while some others may decelerate. The differential equations governing the behavior of the synchronous machines are nonlinear and very complex. Each generator interacts with the others through the transmission network. At some time $t = t_e$, the fault is cleared by opening the circuit breakers. For $t > t_e$, a different set of differential equations will govern the behavior of the system, reflecting the network changes that have taken place at $t = t_e$. The problem of transient stability is thus studied in two steps:

Step A, in which the evolution of the system from t_o to t_e is studied (faulted state); and

Step B, in which the evolution of the system for
t > t_e is studied (post-fault state).

If t_e < t_c, then the system is stable (i.e., it will
not lose synchronism), and if t_e > t_c, then the system is
unstable (i.e., it will lose synchronism). Without loss
of generality, we assume that t_o = 0. The load flow of
the faulted power system and the internal voltages for the
pre-fault power system are listed in Table IV and Table V,
respectively.

The differential equation $\dot{X}$ = $F_A(X)$ of the faulted
state is solved during the period 0 < t < t_e using the
equations (14)-(17) and the data from Tables I, IV, and
V. The initial values X(0) of the state variable
X^T= (δ_1 w_1 δ_2 w_2 δ_3 w_3 δ_4 w_4) are $X(0)^T$ = ($\delta_1(0)$, 0,
$\delta_2(0)$, 0, $\delta_3(0)$, 0, $\delta_4(0)$, 0), where the values of
$\delta_i(0)$, i = 1, 2, 3, 4 are taken from Table V. It is
obvious that this system $\dot{x}$ = $F_A(X)$ is unstable.
Therefore, switching from this system to the other system
(i.e., the post-fault system) by opening the circuit
breaker is necessary at time t=t_e. Next, using the data
of Tables I, II and III, the differential equation
$\dot{Y}$ = $F_B(Y)$ of the post-fault state is solved for t > t_e.
Note that the initial values of the post-fault system are

Table IV. Reduced Bus Admittance Matrix $Y_{ij} \angle \theta_{ij}$ for
Faulted System

	1	2	3	4
1	$0.98 \angle -88.3°$	0	$0.044 \angle 81.0°$	$0.581 \angle 90.6°$
2	0	$2.0 \angle -90°$	0	0
3	$0.044 \angle 81.0°$	0	$1.354 \angle -80.8°$	$0.533 \angle 97.9°$
4	$0.581 \angle 90.6°$	0	$0.533 \angle 97.9°$	$2.9 \angle -69.6°$

Table V. Internal Voltage of Pre-fault System

Generator Number	$E_i \angle \delta_i$
1	$1.057 \angle 5.30°$
2	$1.155 \angle 11.08°$
3	$1.095 \angle 5.48°$
4	$1.000 \angle 0.08°$

the final values of the faulted system, i.e.,

$Y(t_e) = X(t_e)$. Using a uniform time step of $\Delta t = 0.02$

seconds, and applying the subroutine ODES (a program in

VAX computer to solve differential equations) to both

systems, it is found that the (actual) fault clearing time

is $t_c = 2.02$ seconds for this power system. Thus, if the

switch were opened at $t = 2.04$, then machine 2 would lose

its synchronism.

Now the critical clearing time is estimated, using

the Lyapunov function approach discussed earlier. From

the table of the solutions of the faulted system

$\dot{X} = F_A(X)$ the norms $\|z_i\|_i$, $i = 1, 2, 3$ are computed at

each step, until the largest estimated time t_c is

obtained, where the following constraint must be

satisfied:

$$\sum_{i=1}^{3} \alpha_i^* \|z_i\|_i < V^o.$$

Using Criterion I for α^*, with $V^o = 0.4023$, we compute

$$\sum_{i=1}^{3} \alpha_i^* \|z_i\|_i = 0.4019 \quad \text{at } t=2.00$$

$$\sum_{i=1}^{3} \alpha_i^* \|z_i\|_i = 0.4057 \quad \text{at } t = 2.02.$$

Thus, the critical clearing time is estimated as $t_c = 2.0$ seconds. Using Criterion II for α^*, with $V^o = 0.47139$, we compute

$$\sum_{i=1}^{3} \alpha_i^* \|z_i\|_i = 0.4689 \quad \text{at } t = 1.98$$

$$\sum_{i=1}^{3} \alpha_i^* \|z_i\|_i = 0.4735 \quad \text{at } t = 2.00$$

$$\sum_{i=1}^{3} \alpha_i^* \|z_i\|_i = 0.4781 \quad \text{at } t = 2.02 .$$

Thus, the critical clearing time is estimated as $t_c = 1.98$ seconds for this case.

VII. APPLICATION TO A TRAIN CONTROL SYSTEM

Using the results of Chapter IV, we will find a
feedback matrix of a linearly interconnected system which
is known to be unstable, such that the equilibrium of the
overall system must be globally asymptotically stable.
The feedback gains should be as small as possible, but, at
the same time, the overall sysem must be globally
asymptotically stable.

Long freight trains, consisting of 100-150 identical
large capacity cars, have been introduced for hauling coal
from mines in the Rocky Mountains to shipping facilities
in the West. The handling of these heavy loads on steep
grades requires multilocomotive traction. Juggling the
train schedules and staffing the trains are only a few
economical reasons for running long freight trains instead
of several shorter ones.

The train is approximated by a nonlinear mass-spring
dashpot model for longitudinal dynamics and is described
in [16] in detail. A part of the modeled string is shown
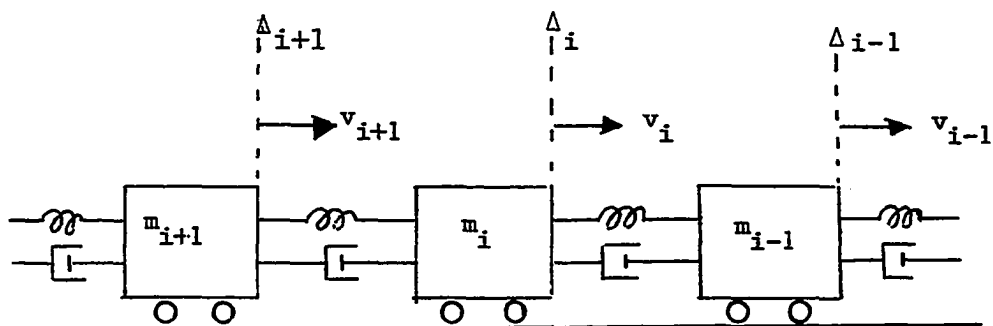in Figure 5.

Figure 5. String of modeled train vehicles

Several simplifications have been made for the control design [6]:

1) The model is linearized to a time invariant system for a certain speed range.

2) The train configuration is chosen symmetric with a locomotive at each end and one in the middle.

3) The mass of each car is m_c and that of each locomotive is m_L.

4) The couplers are modeled with a spring constant k and the damping coefficients c_c for a car and c_L for a locomotive. The deadzones are neglected.

5) Power constraints on the brakes and throttles will not be considered here.

If $\underline{v}$ represents the actual velocity vector and $\underline{f}$ represents the actual input vector consisting of throttling and braking forces, then the deviations from nominal values are defined by

$$\delta \underline{v} = \underline{v} - \underline{v}^o \text{ and } \delta \underline{f} = \underline{f} - \underline{f}^o,$$

where the nominal velocity $\underline{v}^o$ is maintained by $\underline{f}^o$, which is equal to the resistance and gravity forces. The linearized model is then given by the following set of equations [6]:

$$m_i \delta \dot{v}_i = k(\Delta_i - \Delta_{i+1}) + c_i(\delta v_{i-1} - 2\delta v_i + \delta v_{i+1}) + \delta f_i \tag{20}$$

for $i = 1, \cdots, n$, and

$$\dot{\Delta}_i = \delta v_{i-1} - \delta v_i \tag{21}$$

for $i = 2, \cdots, n$,

where Δ_i is the displacement from the nominal force position of the car with $\Delta_1 = \Delta_{n+1} = 0$;

δv_i is the velocity deviation with $\delta v_o = \delta v_{n+1} = 0$;

δf_i is the feedback input;

m_i is the mass of the train members;

$m_i = m_L$ and $c_i = c_L$ for $i = 1$, m, n; and

$m_i = m_C$ and $c_i = c_C$ for $i = 2$, $\cdots$, $m-1$, $m+1$, $\cdots$, $n-1$

with m = the middle locomotive position and

n = the tail locomotive position.

If we let $x_i = \Delta_{i+1}$, $i = 1$, $\cdots$, $n-1$, and $y_i = \delta v_i$, $i = 1$, $\cdots$, n, then equations (20) - (21) will be of the form:

$$\dot{z}_i = A_i z_i + A_{i,i-1} z_{i-1} + A_{i,i+1} z_{i+1} + B u_i \qquad (\Sigma_i)$$

for $i=1$, $\cdots$, n, where

$$z_i^T = (x_i, y_i) \text{ for } i=1, \cdots, n-1;$$

$$z_n^T = z_n = y_n;$$

$$A_{1,0} = A_{n,n+1} = 0; \quad B^T = (0 \quad 1); \text{ and}$$

$$u_i = \delta f_i .$$

Let $x^T = (z_1^T, \cdots, z_n^T)$. Then, the system (Σ_i), $i=1, \cdots, n$, can be rewritten as

$$\dot{x} = Ax + U, \qquad\qquad (S)$$

where A is the $(2n-1) \times (2n-1)$ matrix of the form

$$
\begin{bmatrix}
A_1 & A_{12} & \cdots & 0 \\
A_{21} & A_2 & & \\
& & & \vdots \\
\vdots & & & \\
0 & \cdots & & A_n
\end{bmatrix}
$$

and $U^T = (u_1 \cdots u_n)$.

A linear control law of the form

$$U = Kx,$$

where K is the feedback matrix with dimension (2n-1, 2n-1),

is chosen, such that the equilibrium x=0 of the control

loop

$$\dot{x} = (A + K)x$$

is asymptotically stable.

For a simple string of three vehicles, an optimal

linear feedback is found in [17], which requires that the

force acting on each vehicle be a function of all position

and velocity deviations. Thus, in case of 50 vehicles,

for example, there are 49 position deviation variables and

50 velocity deviation variables. Thus, there are 99

inputs to the control system of each vehicle in the

string, or a total of 4950 (=99x50) links in the overall

system. It seems unreasonable to expect that the position

of the first vehicle will have much effect on the position

of the last vehicle in a long string.

In [6], a 63-member train was simulated along a flat-

down-hill and flat-uphill section with 2 percent slopes.

Since the complexity of this problem does not allow

straightforward evaluation of the feedback matrix K, the

authors of [6] use physical reasoning to make several

simplifications in the original problem. Then, they find

a suboptimal switching sequence for the simplified model,
and check the stability of the overall system by
simulation.

The feedback matrix K is found by assuming that only
the state variable z_i of the isolated subsytem (S_i) is
available for feedback in the subsystem (Σ_i), i.e., the
feedback input is of the form $u_i = K_i z_i$, $i=1, \cdots, n$.
Thus, the matrix K will be of the form

$$
K = \begin{bmatrix}
K_1 & & & & 0 \\
& K_2 & & & \\
& & \bullet & & \\
& & & \bullet & \\
0 & & & & K_n
\end{bmatrix}.
$$

The aggregation method, as suggested in [6], is used to
find the matrix K to insure that the system with feedback
is globally asymptotically stable.

The diagonal dominance property of an M-matrix is
used in testing the aggregated matrix $D = [d_{ij}]$ of Theorem
5 in Chapter IV, i.e., with $d_{ii} > 0$ and $d_{ij} < 0$, $i \neq j$, we have

$$
d_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^{n} |d_{ij}|.
$$

To simplify the problem, we consider the case of a relatively short train consisting of 3 locomotives and 4 cars. We assume the following [17]:

the locomotive mass $m_L = 2$;

the car mass $m_c = 1$;

the damping coefficients $c_L = c_c = 1$; and

the spring constant $k = 1$.

Then, the interconnected systems are of the form

$$\dot{z}_i = A_i z_i + A_{i,i-1} z_{i-1} + A_{i,i+1} z_{i+1} + u_i, \quad u_i = K_i z_i \qquad (\Sigma_i)$$

for $i = 1, \cdots, 6$, where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.5 \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 0 & -1 \\ 0 & 0.5 \end{bmatrix},$$

$$A_2 = A_3 = A_5 = A_6 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix},$$

$$A_{21} = A_{32} = A_{54} = A_{65} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

$$A_{23} = A_{34} = A_{56} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0 \\ -0.5 & -1 \end{bmatrix}, \quad A_{43} = \begin{bmatrix} 0 & 0 \\ 0.5 & 0.5 \end{bmatrix}, \quad A_{45} = \begin{bmatrix} 0 & -1 \\ 0 & 0.5 \end{bmatrix},$$

$$A_{67} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } A_{10} = 0.$$

For i=7, $z_i = a_i z_i + b^T z_{i-1} + u_i$, $\quad(\Sigma_i)$

where $u_i = K_i z_i$, $a_i = -0.5$, and $b^T = (0.5, 0.5)$.

Let (S_i) be the isolated subsystem of (Σ_i) as below:

$$\dot{z}_i = A_i z_i + K_i z_i \triangleq J_i z_i \quad\quad (S_i)$$

for i=1, $\cdots$, 6, where

$$A_i = \begin{bmatrix} 0 & 1 \\ -a_{i1} & -a_{i2} \end{bmatrix} \text{ with } a_{i1} > 0 \text{ and } a_{i2} > 0.$$

Let $J_i = \begin{bmatrix} 0 & 1 \\ -\alpha_i & -\beta_i \end{bmatrix}$ with $\alpha_i > 0$ and $\beta_i > 0$, and let

$K_i = \begin{bmatrix} 0 & 0 \\ -k_{i1} & -k_{i2} \end{bmatrix}$ with $k_{i1} > 0$ and $k_{i2} > 0$, such that

$k_{i1} = \alpha_i - a_{i1}$ and $k_{i2} = \beta_i - a_{i2}$.

The α_i and β_i in J_i are selected such that the gains,

k_{i1} and k_{i2} , of the feedback matrices K_i are as small as possible and such that the diagonal element absolutely dominates the sum of the absolute values of the off-diagonal elements in the i-th row, i=1, ···,n of the test matrix of Theorem 5.

In order for the isolated subsystem (S_i) to be asymptotically stable, the real parts of all the eigenvalues $\lambda(J_i)$ must be negative. From the determinant of $(\lambda I - J_i) = \lambda^2 + \beta_i\lambda + \alpha_i = 0$, with $\alpha_i > 0$ and $\beta_i > 0$, the

locus of $\lambda(J_i) = \dfrac{-\beta_i \pm \sqrt{\beta_i^2 - 4\alpha_i}}{2}$ is shown in Figure 6.

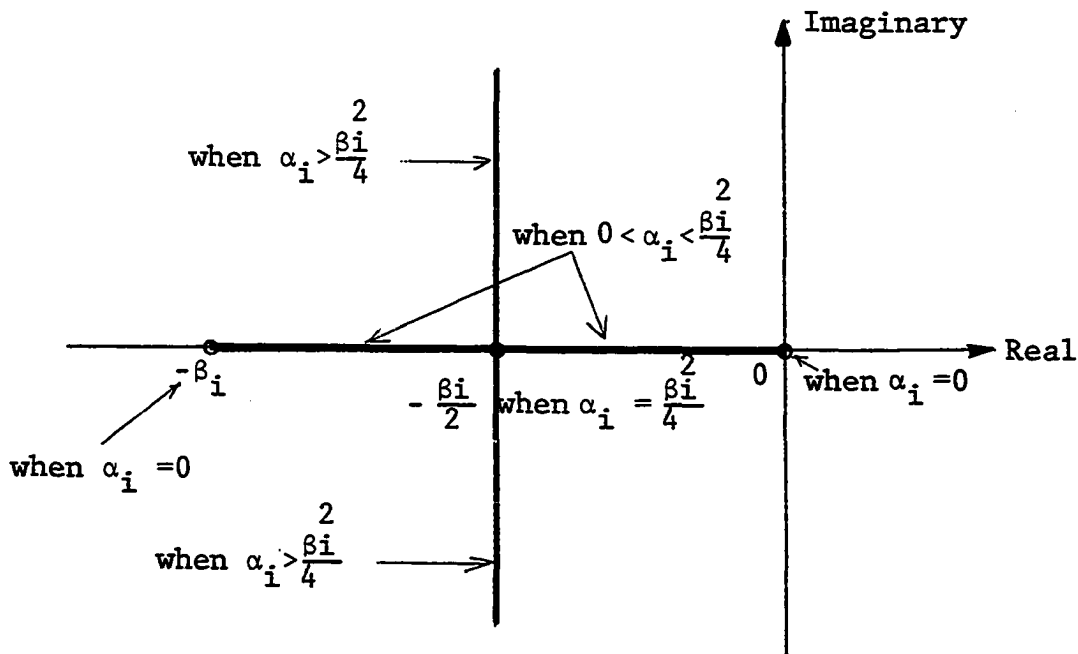For the matrix $M_i = I + h_i J_i$, $h_i > 0$ is chosen such

Figure 6. Root Locus of $\lambda(J_i)$

that M_i is asymptotically stable, i.e., $|\lambda(M_i)| < 1$, and the eigenvalues of M_i are expressed as

$$\lambda(M_i) = \frac{2 - \beta_i h_i \pm h_i \sqrt{\beta_i^2 - 4\alpha_i}}{2}.$$

From the expressions of $\lambda(J_i)$ and $\lambda(M_i)$, we note that

$$\lambda(M_i) = 1 + h_i \lambda(J_i)$$

and that if α_i approaches 0, then $\lambda(M_i)$ approaches the unit circle.

The stability measure μ_i of the isolated subsystem (S_i) is closely related to β_i and is dependent on the choice of the value for h_i. In the present case, β_i corresponds to the damping coefficient of the differential equation $\ddot{y} + \beta_i \dot{y} + \alpha_i y = 0$.

In Lemma 2, which follows, we show that μ_i is less than $\frac{\beta_i}{2}$.

Lemma 2. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleq Jx, \qquad (I)$$

where $x^T = (x_1, x_2)$, $J = \begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix}$, $\alpha > 0$, and $\beta > 0$.

Suppose, for $h > 0$ sufficiently small, there exists $k > 0$ such that $|\lambda(M)| \leqslant k < 1$, where $M = I + hJ$. Then, the stability measure μ of the system (I), where $\mu = (1 - \frac{1}{\rho}) (\frac{1}{h}) > 0$ for some $\rho > 1$, has the upper bound, as follows:

For $0 < \alpha < \frac{\beta^2}{4}$, $\mu \leqslant \frac{\beta}{2} - \frac{\sqrt{\beta^2 - 4\alpha}}{2}$.

For $\alpha > \frac{\beta^2}{4}$, $\mu < \frac{\beta}{2}$.

Proof: From [1], we note that matrix A is stable if and only if $|\lambda(A)| \leqslant 1$ and the eigenvalues of A on the unit circle are simple. We have the eigenvalues of

$\rho M = \rho \begin{bmatrix} 1 & h \\ -\alpha h & 1-\beta h \end{bmatrix}$ as below:

$$\lambda(\rho M) = \rho \lambda(M) = \rho[1 + h\lambda(J)], \qquad (21)$$

where $\lambda(J) = \dfrac{-\beta \pm \sqrt{\beta^2 - 4\alpha}}{2}$.

Case I: For $0 < \alpha < \frac{\beta^2}{4}$, we have the real eigenvalues $\lambda(J) < 0$. We assume $\lambda(M) = 1 + h\lambda(J) > 0$ for sufficiently small $h > 0$. Since M has two simple eigenvalues, the value

of $h > 0$ is chosen such that the maximum value, r, of two

eigenvalues $\lambda(M) = 1 + h \dfrac{-\beta \pm \sqrt{\beta^2 - 4\alpha}}{2}$ is less than one,

i.e.,

$$0 < r = 1 + h \frac{-\beta + \sqrt{\beta^2 - 4\alpha}}{2} < 1.$$

From this, we can choose $\rho > 1$ such that $\rho r < 1$

(i.e., ρM is stable). Thus, we have

$$\rho r = \rho \left(1 + h \frac{-\beta + \sqrt{\beta^2 - 4\alpha}}{2}\right) < 1$$

$$1 - \frac{1}{\rho} < -h \frac{-\beta + \sqrt{\beta^2 - 4\alpha}}{2}.$$

So we have

$$\mu = \left(1 - \frac{1}{\rho}\right)\left(\frac{1}{h}\right)$$

$$< - h \frac{-\beta + \sqrt{\beta^2 - 4\alpha}}{2}\left(\frac{1}{h}\right)$$

$$= \frac{\beta}{2} - \frac{\sqrt{\beta^2 - 4\alpha}}{2} \quad \text{(and so } \mu < \frac{\beta}{2}\text{)}.$$

Case II: For $\alpha = \dfrac{\beta^2}{4}$, $\lambda(M) = 1 - h\dfrac{\beta}{2}$ (double roots).

In this case, if we choose $\rho > 1$ such that $|\lambda(\rho M)| = 1$, then

the eigenvalues of ρM on the unit circle are repeated, and

so ρM is not stable. Thus, we choose $\rho > 1$ such that $|\lambda(\rho M)| < 1$. From (21).

$$\rho \left| 1 - h\frac{\beta}{2} \right| < 1.$$

Here, $h > 0$ can be chosen such that $1 - h\frac{\beta}{2} > 0$.

$$1 - \frac{1}{\rho} < 1 - (1 - h\frac{\beta}{2}) = \frac{\beta}{2}h.$$

Then, $\mu = (1 - \frac{1}{\rho})(\frac{1}{h})$
$$< \frac{\beta}{2}.$$

Case III: For $\alpha > \frac{\beta^2}{4}$, we have

$$\lambda(J) = \frac{-\beta \pm j\sqrt{4\alpha - \beta^2}}{2}, \text{ where } j = \sqrt{-1}.$$

Since $|\lambda(M)| < k < 1$, $k > 0$, we can choose $\rho > 1$ such that $|\lambda(\rho M)| < 1$. From (21),

$$\rho \left| 1 + h\lambda(J) \right| < 1$$

$$1 - \frac{1}{\rho} < 1 - \left| 1 + h\lambda(J) \right|.$$

Thus, $\mu = (1 - \frac{1}{\rho})(\frac{1}{h})$

$$< \frac{1}{h} - \frac{1}{h} \mid 1 + h\lambda(J) \mid$$

$$= \frac{1}{h} - \frac{1}{h} \mid 1 + h \frac{-\beta \pm j\sqrt{4\alpha - \beta^2}}{2} \mid$$

$$= \frac{1}{h} - \frac{1}{h} \mid (1 - h\frac{\beta}{2}) \pm jh \frac{\sqrt{4\alpha - \beta^2}}{2} \mid$$

The value of $h > 0$ can be chosen so that $1 - h\frac{\beta}{2} > 0$

holds. If we use the inequality $a < \mid a \pm jb \mid$ for

$a = 1 - h\frac{\beta}{2} > 0$ and $b = h\frac{\sqrt{4\alpha - \beta^2}}{2} > 0$ in the above, then

we get

$$\mu < \frac{1}{h} - \frac{1}{h} (1 - h\frac{\beta}{2})$$

$$= \frac{\beta}{2}.$$

Numerical results show that μ_i for $\alpha_i > \frac{\beta_i^2}{4}$ can be

larger than that for $0 < \alpha_i < \frac{\beta_i^2}{4}$. Once β_i is chosen for the

stability measure μ_i of (S_i), we attempt to choose

$\alpha_i = \frac{\beta_i^2}{4}$ for the smallest feedback gain as well as for the

largest μ_i.

The procedure to stabilize the system is as follows:

Step 1. Choose an initial value of $\beta_i > 0$. This β_i

        will give a measure μ_i of stability for (S_i).

Step 2. Compute $\alpha_i = \dfrac{\beta_i^2}{4}$.

Step 3. Find $h_i > 0$ such that $|\lambda(M_i)| < 1$, and find $\rho_i > 1$ such that $\rho_i M_i$ is stable.

Step 4. Find W_i^* by the constructive algorithm and let $v_i(z_i) = \| z_i \|_i$ be the Lyapunov function of (S_i).

Step 5. Compute $\mu_i = (1 - \dfrac{1}{\rho_i})\,(\dfrac{1}{h_i})$ and $g_{ij} = \| A_{ij} \|_{ij}$.

Step 6. Check the condition of diagonal dominance.

(i.e., for a test matrix $D = [d_{ij}]$, $d_{ii} > \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{n} |d_{ij}|$,

with $d_{ii} = \mu_i > 0$ and $d_{ij} = -g_{ij} < 0$, $i \neq j$.)

If the condition is true, then exit. If the condition is false, choose a larger value for β_i and repeat the procedure listed above.

Note that since the i-th subsystem is influenced by both the (i-1)th and the (i+1)th systems, this algorithm cannot avoid this trial and error procedure.

In the subsystem (Σ_7) expressed by the scalar equation $\dot{z}_7 = + a z_7 + b^T z_6$, where $a = a_7 + K_7 < 0$,

$K_7 < 0$, and $b^T = (0.5 \quad 0.5)$, choose the Lyapunov function

$v_7(z_7) = |z_7|$ for the isolated subsystem (S_7):

$\dot{z}_7 = az_7$, where $|z_7|$ denotes the absolute value of z_7.

Then, we have

$$Dv_{7(S_7)}(z_7) = a|z_7|,$$

so that $\mu_7 = -a$. Then, $g_{76} = \|b^T\|$

$= \max \{b^T z_6 : z_6 \varepsilon E(W_6^*)\}$ is computed, and the condition

$\mu_7 > g_{76}$ is tested.

The resulting numerical data are as follows:

	(S_1)	(S_i) $i=2,3,5,6$	(S_4)
β_i	3	5	4.4
α_i	2.25	6.25	4.84
h_i	0.2	0.2	0.25
ρ_i	1.42	1.9	2.22
μ_i	1.478	2.368	2.198
M_i	$\begin{bmatrix} 1 & 0.2 \\ -0.45 & 0.4 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.2 \\ -1.25 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.25 \\ -1.21 & -0.1 \end{bmatrix}$

The final convex sets W_i^*, i=1, $\cdots$, 6 are constructed by

the constructive algorithm discussed in Section III.B,

picking an initial convex and balanced set W_{oi} such that

$$W_{oi} = \{ \ z_i^T = (x_i, \ y_i): \ |x_i| + |y_i| < 1 \},$$

and the Lyapunov functions $v_i(z_i) = \|z_i\|_i$ for (S_i) are

determined from W_i^* . The graphs of W_i^*, i=1, $\cdots$,6 are

shown in Figure 7.

We now compute $g_{ij} = \|A_{ij}\|_{ij}$ using the norms $\|\cdot\|_i$ and

$\|\cdot\|_j$, and the test matrix D= $[d_{ij}]$ is formed, where

$$d_{ij} = \begin{cases} \mu_i & , \ i=j \\ -g_{ij} & , \ i \neq j, \end{cases}$$

as below:

	1	2	3	4	5	6	7
1	1.478	-1.454					
2	-1	2.368	-1.25				
3		-1	2.368	-1.21			
4			-0.5	2.198	-1.25		
5				-1	2.368	-1.25	
6					-1	2.368	-1
7						-0.5	0.6

D =

Since the test matrix D is an M-matrix, the equilibrium

x=0 of the overall system (S) is asymptotically stable by
Theorem 5 in Chapter IV. The feedback matrix K is:

$$
K = \begin{bmatrix}
K_1 & & & & & 0 \\
& K_2 & & & & \\
& & \bullet & & & \\
& & & \bullet & & \\
& & & & \bullet & \\
0 & & & & & K_7
\end{bmatrix}
$$

where $K_1 = \begin{bmatrix} 0 & 0 \\ -1.75 & -2.5 \end{bmatrix}$,

$K_2 = K_3 = K_5 = K_6 = \begin{bmatrix} 0 & 0 \\ -5.25 & -3 \end{bmatrix}$,

$K_4 = \begin{bmatrix} 0 & 0 \\ -4.34 & -3.4 \end{bmatrix}$ and $K_7 = -0.1.$

Thus, using the Lyapunov function approach, a train
control system is stabilized, such that the system is
globally asymptotically stable.
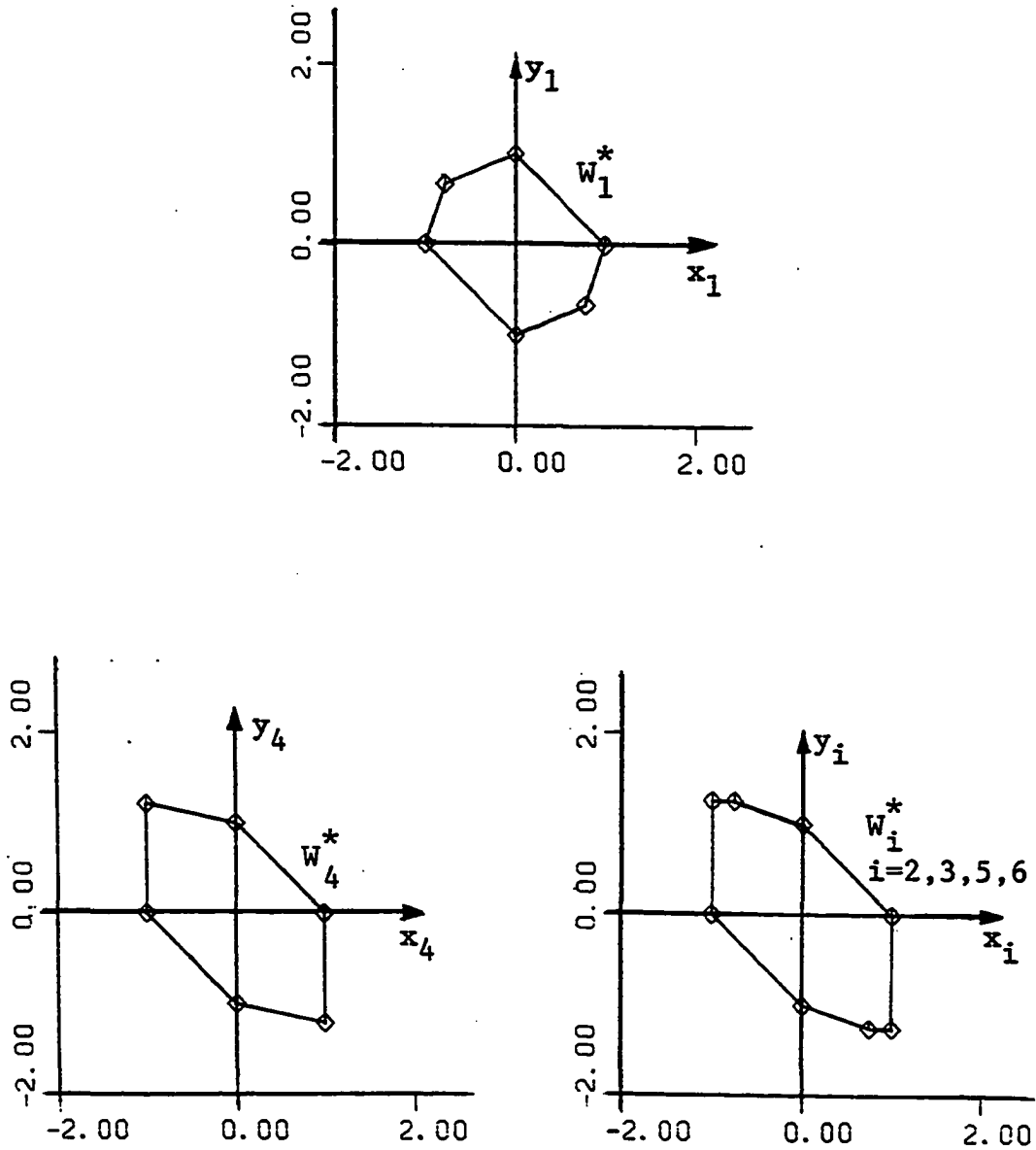
Figure 7. The final convex sets W_i^*, i=1, ...,6

VIII. CONCLUSIONS

In [1] and [2], Brayton and Tong developed a
constructive algorithm as a basis for the stability
analysis of an equilibrium of a dynamical system described
by differential equations of the form $\dot{x}=f(x)$. But this
algorithm is only applicable to a system which is globally
asymptotically stable.

If a system is locally asymptotically stable, then we
are usually interested in seeking the domain of attraction
of an equilibrium of the system. In [3], an efficient
procedure was developed to estimate such a domain of
attraction, using the results of [1] and [2].

Although the above results are significant and
powerful, these results are not easily applied to high
dimensional systems.

In Chapters IV and V, these difficulties are removed
to a certain extent. Large scale systems are viewed as an
interconnection of lower order subsystems. The Lyapunov
functions of the isolated subsystems are generated by the
constructive algorithm developed by Brayton and Tong.
Using these Lyapunov functions, a test matrix for the
stability analysis of the entire system is formed in terms
of the qualitative properties of the isolated subsystems

and in terms of the properties of the system interconnections. If the matrix is an M-matrix, then the equilibrium x=0 of the overall system is asymptotically stable in some region.

In Chapter IV, the region of interest is the whole space R^n, i.e., this chapter addresses the question whether a large scale system is globally asymptotically stable or not.

In Chapter V, where we consider large scale systems with equilibrium points that are locally asymptotically stable, an algorithm is presented to estimate the domain of attraction of an equilibrium. The domain of attraction of the equilibrium of each isolated subsystem of an interconnected system is estimated by using the algorithm in [3]. The domain of attraction for the overall system is estimated using the results of Chapter IV. The Lyapunov function of the overall system is used to compute the largest estimate by summing optimally weighted Lyapunov functions of the isolated subsystems.

The results of Chapter V are applied in the stability analysis of a four generator power system with uniform damping characteristics. The stability region of the equilibrium of the post-fault power system is estimated. Using this Lyapunov function approach, the critical

clearing time is estimated when the system is faulted during its steady-state operation. Numerical results indicate the constructive algorithm is capable of yielding very tight stability bounds.

The results of Chapter IV are applied to a train control system to stabilize the system by feedback. It is shown that the stability measure μ of a system of the form $\ddot{y} + \beta\dot{y} + \alpha y = 0$, $\alpha > 0, \beta > 0$, is less than $\frac{\beta}{2}$. No direct comparisons were made with existing results, since different assumptions were used in [6], [17], than in the present work.

Some suggestions for further research follow.

1. Interconnected power systems with nonuniform damping are more realistic than power systems with uniform damping. In this case, the n-machine power system <u>can't</u> be split into (n-1) subsystems where the dimension of each subsystem is <u>two</u>.

Let the angle and speed differences with respect to a reference machine (say the n-th machine) be

$$\delta_{in} = \delta_i - \delta_n \text{ and } w_{in} = w_i - w_n$$

respectively, and $w_i = \dot{\delta}_i$. Then, a (2n-1) dimensional

state space model can be obtained directly from equations (14)-(17) as

$$\dot{\delta}_{in} = w_{in}$$

$$\dot{w}_{in} = -\lambda_i w_{in} + (\lambda_n - \lambda_i)w_n + M_n^{-1}\sum_{j=1}^{n-1} A_{jn}f_{jn}$$

$$-M_i^{-1}\sum_{\substack{j=1 \\ j \neq i}}^{n} A_{ij}f_{ij} \tag{22}$$

$$\dot{w}_n = -\lambda_n w_n - M_n^{-1}\sum_{j=1}^{n-1} A_{jn}f_{jn}$$

for $i=1,2,\cdots,n-1$, where

$$\lambda_i = D_i/M_i$$

$$A_{ij} = E_i E_j Y_{ij}, \text{ and}$$

$$f_{ij} = \cos(\delta_{ij} - \theta_{ij}) - \cos(\delta_{ij}^o - \theta_{ij}). \tag{23}$$

In (23), δ_{ij}^o's are components of an equilibrium obtained as solutions of the equations

$$P_{ei}(\delta_{ij}^o) = P_{mi}, \quad i=1,\cdots,n-1.$$

The system (22) may be solved if we can find the convex set W^* for the Lyapunov functions of the subsystems in a three dimensional space.

2. Brayton and Tong's constructive algorithm is, by nature, more easily applicable to systems represented by a set of difference equations. Thus, their algorithm may be useful in studying the parameter sensitivity problems and the quantization effects of digital filters.

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X. ACKNOWLEDGMENTS